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# Optimal Policy and Algorithm on A Serial Multiechelon Inventory Model with Secondary Markets

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We study optimal order/shipment and selling policy and effective algorithm computing optimal policy for a periodic-review, serial multiechelon and backlogging inventory model with secondary markets and a finite planning horizon. We demonstrate that when the terminal cost function is a sum of convex functions in echelon inventory positions, the optimal cost functions of all the periods are respectively decomposed into the sums of convex functions in echelon inventory positions. Optimal order/shipment and secondary-market sales are determined recursively as nested sell-down-to and order/ship-up-to policies. By peculiar *stage and interstage optimizations*, we devise efficient algorithms to attain optimal decision rules in each period.

*Key words*: serial multiechelon inventory/distribution systems, multimodulariy, separable convexity
*History*: .

## 1. Introduction

### 1.1. Overview on Series Multiechelon Inventory Models

Cost structures play a vital role in the optimality analysis of inventory and supply chain management literature. It is noticeable that if the procurement cost is proportional, optimal ordering policy for a single-item single-location inventory model is base-stock one; while, if the procurement cost contains a setup cost, optimal ordering policy becomes an $(s, S)$ type. In this paper, we investigate how a structure of terminal cost functions affects optimal ordering/shipment policies for the classical serial multi-echelon inventory model

### 1.2. Our Contributions

Via different approaches, we explore the structures of optimal cost function for the classic series multiechelon inventory systems, optimal policies for order/shipment and down-sales in the secondary markets and draw insights from the structure into the effective behaviors of the optimal policies. We demonstrate that the decomposable terminal cost functions can be recursively preserved by optimal policy for optimal cost functions and propose expedient algorithms to compute optimal order/shipment policies. A great challenge for the series multiechelon inventory models is the permission of satisfying demands at upstream installations since Clark and Scarf (1960). In the current model, demands do occur at the secondary markets near upstream stages and the firm is motivated to sell extra stocks stored at the upstream stages. This raises a trade-off analysis comparing profits from the secondary-market sales and potential profits of moving stocks down along the supply chain. .

We complement the literature in threefold. Firstly, we show that, when the terminal cost function is *additively decomposable* or additively convex (Veinott, Gong and Wang), optimal cost function $f_t(\cdot)$ is also separably convex. Alternatively, as a multi-variable function of echelon inventory positions, $f_t(\cdot)$ can be provably a sum of single-variable convex functions of the echelon inventory positions. Such property greatly simplifies the analysis of the optimal order/shipment decisions and assures the optimality for nested sales in the secondary markets and nested order/shipment policy. Secondly, we adopt a special approach to demonstrate additive decomposability for optimal cost functions and propose efficient algorithms for optimal quantities. Thirdly, we propose a different approach from the penalty cost approach developed in Clark and Scarf (1960) to circumvent complexity risen from the sales of the secondary markets.

## 2. Literature Review

Multi-installation(Clark and Scarf 1960), multi-echelon, multi-stage serial system, multi-location

The study on multi-echelon distribution systems was originated from a seminal work of Clark and Scarf (1960), showing that an echelon base-stock policy is optimal for a periodic-review and finite-horizon serial distribution inventory model. Federgruen and Zipkin (1984) extend this result to infinite horizon and show that a stationary order-up-to-level policy is optimal. Rosling (1989) studies a periodic-review assembly system. Federgruen and Zipkin (1984a, b, c) propose the lower bound for a two-echelon inventory model with one-warehouse and multiple retailers. Chen and Zheng (1994) simplify the optimality results on serial and assembly periodic-review systems. Gallego and Zipkin (1999) provide a more detailed summary of related work and history of development. Shang and Song (2003) develop bounds and heuristic for the classic N-stage serial supply systems with linear costs and stationary random demands in a continuous setting.

We use a different approach from Clark and Scarf (1960) and Chen and Zheng (1994) to show the optimality of the base-stock policy in the classic N-stage serial multi-stage inventory model. In addition, we demonstrate that optimal cost functions are multimodular and can be separable as a sum of convex functions of echelon inventory stocks.

Tim Huh and Janakiraman (2008) consider a periodically reviewed serial system with lost sales and show that optimal value functions are L-natural convex. They derive bounds on the derivative of the optimal order quantities. Tim Huh and Janakiraman (2008) claim that the L-natural convexity is preserved in the serial systems with capacity constraints and/or backordering.

Since Clark and Scarf delivered their seminar work (Clark and Scarf (1960)) on a classic serial multi-echelon inventory model, extensions to and variations of the classic model and numerical procedure have been fruitful and prolific. We consider an extended multi-echelon inventory model with secondary markets (Angelus (2011)). Our analysis can be applied to a classic and periodic serial multi-echelon inventory model with a finite horizon.

## 3. Model Formulation

We consider a serial periodic-review, multi-echelon inventory model with $N$ stages and a planning horizon of $T$ periods. In each period, an exogenous random demand $D$ with density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$ occurs at the lowest or most downstream stage, say stage 1, independently of those demands in future periods. We pursue a *backlogging* setting where inventory deficits at stage 1 are backlogged. In addition, extra on-hand inventory at all the stages can be sold/disposed in their nearby secondary markets, where demands are plentiful and prices $\mathbf{r}_t = (r_{1t}, \cdots, r_{Nt})$ in period $t$ $(t = 1, \cdots, T)$ follow a stochastic process. Sales in the secondary markets remove sold stock immediately from the stages, while replenishment from the secondary markets is not an option at any stage. The whole supply chain however can be replenished solely from an outside supplier at the highest, or most upstream stage, say stage $N$, with a positive leadtime $L_N$ periods. Stage $i$ $(i = 1, \cdots, N-1)$ can be replenished by shipments from its upstream stage $i+1$ with a leadtime $L_i$. Stock deficit is not allowed at all stages. This is a revisit of serial multiechelon inventory model with secondary markets explored in Angelus (2011), except for the formulation of the stochastic prices, which are formulated as a Markov-modulated stochastic process in Angelus (2011). A diagram of the system is shown in Figure 1.

The inventory order/shipment cost at stage $i$ $(i = 1, \cdots, N)$ is proportional to order/shipment quantities with a variable cost $c_i$. At stage $i$ $(i = 1, \cdots, N)$, we denote by $x_i^j$ in-transit inventory from stage $i+1$, where stage $N+1$ is regarded as that outside supplier with ample stocks, to stage $i$ $(j = 1, \cdots, L_i - 1)$, which will arrive due in $j$ periods, and by $x_i^0$ on-hand stock at stage $i$. The inventories in transit and on hand are compacted into $\mathbf{x}_i = (x_i^0, \cdots, x_i^{L_i-1})$ as an $L_i$-vector of

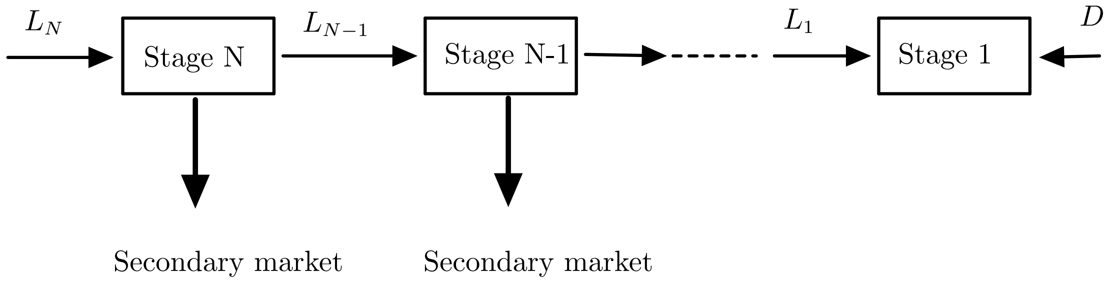


**Figure 1 The Diagram of the Multi-echelon System with the Secondary Markets**

pipeline inventories at stage $i$ $(i=1,\cdots,N)$. Holding inventories $(x_i^0, x_{i-1}^{L_{i-1}-1},\cdots,x_{i-1}^1)$ at stage $i$ $(i=2,\cdots,N)$ incurs a unit stage (installation) holding cost $\hat{h}_i$ dollars per unit per period. It is customary to assume that $\hat{h}_N<\cdots<\hat{h}_1$, because holding a product unit at lower stages is more expensive. Traditionally, unit echelon holding costs at all the stages are counted by $h_i=\hat{h}_i-\hat{h}_{i+1}$ at stage $i$ $(i=1,\cdots,N)$ where $\hat{h}_{N+1}=0$. Unit penalty cost $p$ is accrued at stage 1 for backlogs. In each period, we place an order/shipment for stage $i$ $(i=1,2,\cdots,N)$, $q_i$, to stage $i+1$ and sell $z_i$ units at stage $i$ $(i=1,\cdots,N)$ to the secondary market from extra on-hand stocks at the stage. $\beta\in[0,1]$ is a discount factor. Price process in the secondary markets along the whole supply chain, $\{\mathbf{r}_t|t=1,\cdots,N\}$, evolves as a discrete-time Markov chain, where $\mathbf{r}_t=(r_{1t},\cdots,r_{Nt})$ is an $N$-vector, reflecting market speculative volatility and competitive shocks. Like Markov-modulated price chain in Angelus (2011) that dominates prices at all the stages, our price process is exogenous and independent of demands at stage 1 and decisions at the stages. It has become common with the growth of online and offline divisions among the supply chains, and from existing commodity competitive markets, that this sort of price processes are extensively assumed in the literature.

The events in any period are sequenced as follows: Upon the receipt of all the in-transit inventories due in this period at their stages, all the on-hand and in-transit inventories at the stages, and prices at the secondary markets are reviewed. Each stage then places an order at its adjacent upstream stage and sells extra on-hand stock to the nearby secondary market. Demand occurs at stage 1 to delete available on-hand inventory at the stage and its unfilled amount is backlogged. At the end of the period, the holding and penalty costs are incurred in terms of the inventories of the stages.

Order, shipment, and selling decisions at all the relevant stages are centrally made by the system manager exclusively and recursively to minimize the expected present value of a total discounted cost over the planning horizon. The problem is therefore analyzed by a stochastic dynamic programming, in order to optimize ordering and selling decisions at all the stages recursively over periods.

In the context of the dynamic programming, state dynamic cross-period transitions should be specified primarily. In period $t$ $(t=1,\cdots,T)$, when orders $(q_1,\cdots,q_N)$ are placed, extra on-hand stocks $(z_1,\cdots,z_N)$ are sold to the $N$ secondary markets, and a demand $D$ comes at stage 1, state $(\mathbf{x}_1,\cdots,\mathbf{x}_N)$ will be transferred to and observed at the beginning of period $t+1$ as follows: At stage $i$ $(i=2,\cdots,N)$, $\bar{\mathbf{x}}_i=(x_i^1+x_i^0-q_{i-1}-z_i,x_i^2,\cdots,x_i^{L_i-1},q_i)$, and at stage 1, $\bar{\mathbf{x}}_1=(x_1^0+x_1^1-D,x_1^2,\cdots,x_1^{L_1-1},q_1)$. We denote by $f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N)$ optimal cost function starting from state $(\mathbf{x}_1,\cdots,\mathbf{x}_N)$ in period $t$ and recursively compute it by optimality equations:

$$\begin{aligned}&f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)\\ =&\min_{\substack{i\in\{1,\cdots,N\},x_{i+1}^0\geq q_i+z_{i+1}\geq 0\\ q_i,z_{i+1}\geq 0,x_{N+1}^0=\infty,z_{N+1}=0}}\left\{\begin{array}{c}\sum_{i=1}^N c_iq_i-\sum_{i=2}^N r_{it}z_i+\sum_{i=2}^N \hat{h}_i(\sum_{j=1}^{L_{i-1}-1}x_{i-1}^j+x_i^0+q_i-z_i-q_{i-1})\\ +\hat{h}_1E(x_1^0-D)^++pE(x_1^0-D)^-+\beta E[f_{t+1}(\bar{\mathbf{x}}_1,\cdots,\bar{\mathbf{x}}_N),\mathbf{r}_{t+1})|\mathbf{r}_t]\end{array}\right\},\end{aligned}\tag{3.1}$$

where $f_{T+1}(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)=v_T(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)$. The goal of (3.1) is to minimize a sum of ordering costs in the first term, the sales revenue (negative cost) on the secondary markets in the second term, the holding costs of the on-hand and in-transit stocks at the stages all but stage 1 in the third term, the expected holding and shortage costs at stage 1 in the fourth and fifth terms, respectively, and the discounted costs incurred in period $t+1$ and beyond in the last term.

Optimality equations (3.1) is named as *stage formulation* by Alexander Angelus (2011), based on stage variables $(\mathbf{x}_1,\cdots,\mathbf{x}_N)$. (3.1) can also be written in the echelon inventory positions later for convenience.

## 4. Decomposable Optimal Cost Functions

In this section, we reveal that echelon inventory positions are complementary and substitutable for the current serial multiechelon inventory model and thus, optimal cost functions are decomposable by (Topkis 1998) in the echelon inventory positions.

### 4.1. Complementary Echelon Inventory Positions

Unlike the classic serial multiechelon inventory/distribution systems (Clark and Scarf 1960), demands in the current model also occur at upstream stages through the secondary markets. In most of single-item inventory and supply chain applications (Zipkin 2008a, Li and Yu 2014), inventory positions have been demonstrated to be complementary in the literature. However, in the current system of more than one demands, whether echelon inventory positions are complementary is just a conjecture, and should be confirmed strictly.

In line with a state $(\mathbf{x}_1, \cdots, \mathbf{x}_N)$ of stage variables, echelon inventory positions can be formed customarily: At stage $i$ $(i=1,\cdots,N)$, the $k$-th echelon inventory position $(k=0,\cdots,L_i-1)$ is the sums of inventories in-transit and on-hand at downstream stages and those that will be available by $j$ periods at the stage, such as,

$$y_1^0 = x_1^0, \cdots, y_1^l = \sum_{j=0}^{l} x_1^j, \cdots, y_1^{L_1-1} = \sum_{j=0}^{L_1-1} x_1^j, \cdots,$$
$$y_l^k = \sum_{i=1}^{l-1} \sum_{j=0}^{L_i-1} x_i^j + \sum_{j=0}^{k} x_l^j, \cdots, y_N^{L_N-1} = \sum_{i=1}^{N-1} \sum_{j=0}^{L_i-1} x_i^j + \sum_{j=0}^{L_N-1} x_N^j.$$

This linear transformation that converts the stage variables into the echelon inventory positions and vice versa is invertible. Clearly, optimal cost functions can be represented by either of the stage variables or the inventory positions. In particiular, via this invertible transformation,

$$f_t(\mathbf{x}_1, \cdots, \mathbf{x}_N, \mathbf{r}_t) = \Psi_t(0, \mathbf{y}_1, \cdots, \mathbf{y}_N, \mathbf{r}_t)$$

is a function of the echelon inventory positions, $\Psi_t(0, \mathbf{y}_1, \cdots, \mathbf{y}_N, \mathbf{r}_t)$, where prices $\mathbf{r}_t$ in the secondary markets are identical. For a technicality, echelon inventory positions, $(\mathbf{y}_1, \cdots, \mathbf{y}_N)$, must comply with $y_i^j \le y_i^{j+1}$, $y_i^{L_i-1} \le y_{i+1}^0$ for $i=1,\cdots,N$ and $j=0,1,\cdots,L_i-2$, where $y_{N+1}^0=\infty$ conventionally.

These inventory positions provide resources (finished goods) from the suppliers to satisfy demand and sustain the system effectively. When the system is run optimally, the demand and supply should be balanced economically and the inventory positions are *complementary*. Therefore, optimal cost function $f_t(\cdot)=\Psi_t(0,\cdot)$ can be thought of as a *submodular functions* in terms of the echelon inventory positions. As minimizing $f_t(\mathbf{x}_1, \cdots, \mathbf{x}_N, \mathbf{r}_t)$ is equivalent to minimizing $\Psi_t(0, \mathbf{y}_1, \cdots, \mathbf{y}_N, \mathbf{r}_t)$, when the echelon inventory positions are provably complementary, we can adopt the *complementarity analysis* (Topkis 1998) to expedite exploration for optimal inventory replenishment and sales control on this serial multi-echelon inventory model.

When a function of the stage variables is submodular in the echelon inventory positions, it is multimodular by a later formal definition in the stage variables. Interestingly, multimodular functions are critical characteristics for many single-item inventory systems with complementary inventory positions (Zipkin 2008a, Li and Yu 2014). However, as submodular functions must be defined carefully in sublattices (Topkis 1998), and current optimal cost functions are multimodular functions in the stage variables and submodular functions in the echelon inventory positions, it is required to choose appropriate feasible sets for the stage variables. With this regard, we select special polyhedrons, called *multimodular polyhedrons* for optional domains. A multimodular polyhedron $\mathcal{V} \subseteq \mathbb{R}^n$ such as

$$\mathcal{V} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a}_i \cdot \mathbf{x} \ge b_i, i=1,\cdots,K\}$$

is constituted with $K$ linear constraints, where, for $i=1,\cdots,K$, the non-zero entries of $\mathbf{a}_i$ locate successively and are either entirely 1 or entirely -1, $b_i\in\mathbb{R}$, and $\mathbf{a}_i\cdot\mathbf{x}=\sum_{j=1}^n a_i^j x_j$ for $\mathbf{x}\in\mathbb{R}^n$. According to Example 2.2.7 (b) in (Topkis 1998), a multimodular polyhedron $\mathcal{V}$ can induce a sublattice of $\mathbb{R}^n\times\mathbb{R}$, such as

$$\mathcal{S}=\{(\mathbf{y},\eta)\in\mathbb{R}^n\times\mathbb{R}:(y_0-\eta,y_1-y_0,\cdots,y_{n-1}-y_{n-2})\in\mathcal{V}\}.$$

As such $\mathcal{S}$ is a sublattice, it can provide domain for submodular functions. Thus, a multimodular polyhedron $\mathcal{V}$ can host a multimodular function $f:\mathcal{V}\to\mathbb{R}$ if $\Psi(\eta,\mathbf{y})=f(y_0-\eta,y_1-y_0,\cdots,y_{n-1}-y_{n-2})$ is submodular on $\mathcal{S}$, where $\eta$ is an arbitrary real number. It is worthwhile to point out that $\Psi(\eta,\mathbf{y})=\Psi(\eta+a,\mathbf{y}+a\mathbf{e}^n)$ for any real $a$ where $\mathbf{e}^n$ is an $n$-vector of ones. For more about multimodular functions based on this general definition, we refer readers to Li and Yu (2014), and Murota (2003) and the references therein. The multimodular functions of the stage variables lead to the special $L^\natural$-convex functions of the echelon inventory positions. The $L^\natural$-convex functions are introduced early by Murota (2005), and later employed by Zipkin (2008a) to resolve some inventory problems.

In terms of this general definition, if the echelon inventory positions are complementary, optimal cost function $f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N)$ is multimodular. With additional $\eta$, the stage variables can be transferred by an invertible unimodular transformation into new complementary variables $(\mathbf{u}_1,\cdots,\mathbf{u}_N)$ and arbitrary $\eta$, such that, $x_i^j=u_i^j-u_i^{j-1}$ $(i=1,\cdots,N,\ j=1,\cdots,L_i-1)$, $x_i^0=u_i^0-u_{i-1}^{L_{i-1}-1}$, and $u_0^{L_0-1}=\eta$. Using $\eta=u_0^{L_0-1}$, these new variables are actually the echelon inventory positions plus $\eta$: $u_i^j=y_i^j+\eta$, $(i=1,\cdots,N,j=0,\cdots,L_i-1)$. Thereby, the optimal multimodualr cost function of period $t$, $f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N)$, of the stage variables can be converted into,

$$f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)=\Psi_t(\eta,\mathbf{u}_1,\cdots,\mathbf{u}_N,\mathbf{r}_t),$$

where $\mathbf{u}_i=\mathbf{y}_i+\eta\mathbf{e}^{L_i}$ $(i=1,\cdots,N)$, which is a submodular function of the sums of the inventory positions and $\eta$.

Let $d$ represent a realization of $D$, order placements $(q_1,\cdots,q_N)$ and selling quantities $(z_1,\cdots,z_N)$, respectively. The echelon inventory positions in a period are transferred by them into

$$\begin{aligned}
\bar{y}_1^0 &= y_1^1-d-z_1,\cdots,\bar{y}_1^{L_1-1}=y_1^{L_1-1}+q_1-d-z_1,\\
&\cdots,\\
\bar{y}_l^0 &= y_l^1-d-\sum_{j=1}^{l}z_j,\cdots,\bar{y}_l^{L_l-1}=y_l^{L_l-1}-d-\sum_{j=1}^{l}z_j+q_l,\\
&\cdots,\\
\bar{y}_N^0 &= y_N^1-d-\sum_{j=1}^{N}z_j,\cdots,\bar{y}_N^{L_N-1}=y_N^{L_N-1}-d-\sum_{j=1}^{N}z_j+q_N,
\end{aligned}$$

which will appear at the beginning of next period as the echelon inventory positions. Therefore, optimality equations (3.1) can be written in terms of the echelon inventory positions in line with the preceding cross-period evolutions. The optimality equations in terms of the echelon inventory positions are *echelon formulation* called in Angelus (2011). We will present echelon formulation in (4.4).

### 4.2. Multimodularity Preservation

To entail that the decomposability of a decomposable terminal cost function can be relayed backward recursively by optimal order/shipment and sales decisions, regardless of random competitive prices in the secondary markets, we must demonstrate that optimal cost functions are multimodular in the stage variables. We assume that a terminal cost function $v_T(\mathbf{x}_1,\cdots,\mathbf{x}_N)$ is a sum of convex functions in the echelon inventory positions: for any price vector $\mathbf{r}_t$,

$$v_T(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)=f_{T+1}(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)=\Psi_{T+1}(0,\mathbf{y}_1,\cdots,\mathbf{y}_N,\mathbf{r}_t)=\sum_{i=1}^{N}\sum_{j=0}^{L_i-1}\psi_{ij,T+1}(y_i^j,\mathbf{r}_t). \quad (4.2)$$

One such a terminal function can be formed by moving all the stocks forward to stage 1 after the end of the planning horizon for the fulfillment of cumulative demands over $\sum_{i=1}^{N}L_i$ periods and salvaging leftovers at the holding price. It can be expressed by

$$\Psi_{T+1}(0,\mathbf{y}_1,\cdots,\mathbf{y}_N,\mathbf{r}_t)=\sum_{i=1}^{N}\sum_{j=0}^{L_i-1}\beta^{n_{ij}}EL(y_i^j-\sum_{l=1}^{i-1}\sum_{k=0}^{L_l-1}D_{lk}-\sum_{k=0}^{j}D_{ik}), \quad (4.3)$$

which, with $n_{ij}=\sum_{l=1}^{i-1}L_l+j\ (i=1,\cdots,N,j=1,\cdots,L_i)$, is a decomposable terminal cost function of (4.2).

As $\hat{h}_i>\hat{h}_{i+1}$ for $i=1,\cdots,N$, where $\hat{h}_{N+1}=0$, it is convenient to replace stage (installation) holding costs into echelon holding costs such as $h_i=\hat{h}_i-\hat{h}_{i+1}, i=1,\cdots,N$. Consequently, immediate costs in a period can be rewritten in optimality equations (3.1) in terms of the echelon inventory positions:

$$\begin{aligned}&\sum_{i=1}^{N}c_iq_i-\sum_{i=1}^{N}r_iz_i+\sum_{i=2}^{N}\hat{h}_i(\sum_{j=1}^{L_{i-1}-1}x_{i-1}^j+x_i^0+q_i-z_i-q_{i-1})+E\hat{h}_1(x_1^0-z_1-D)^++bE(x_1^0-z_1-D)^-\\=&\sum_{i=1}^{N}c_i(y_i^{L_i}-y_i^{L_i-1})-\sum_{i=1}^{N}(r_i+\sum_{k=i}^{N}h_k)z_i+\sum_{i=1}^{N}h_i(y_i^0-ED)+(b+\sum_{i=1}^{N}h_i)E(y_1^0-z_1-D)^-\end{aligned}$$

where for $i=1,\cdots,N$, $y_i^{L_i}=y_i^{L_i-1}+q_i$ is the post-inventory position at stage $i$.

With this equation, optimality equation (3.1) can be changed from the stage formulation into the *echelon formulation* in terms of the echelon inventory positions,

$$f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)=\min_{\substack{y_{i+1}^0\geq y_i^{L_i}+z_{i+1},y_i^{L_i}\geq y_i^{L_i-1},\\ z_{i+1}\geq 0,i\in\{1,\cdots,N\},y_N^{L_N}\geq y_N^{L_N-1}}}\left\{\begin{array}{c}\sum_{i=1}^N c_i(y_i^{L_i}-y_i^{L_i-1})-\sum_{i=1}^N(r_i+\sum_{k=i}^N h_k)z_i\\ +\sum_{i=1}^N h_i(y_i^0-ED)+(b+\sum_{i=1}^N h_i)E(y_1^0-z_1-D)^-\\ +\beta E[\Psi_{t+1}(0,\bar{\mathbf{y}}_1,\cdots,\bar{\mathbf{y}}_N,\mathbf{r}_{t+1})|\mathbf{r}_t]\end{array}\right\} \tag{4.4}$$

where $E[\Psi_{t+1}(0,\bar{\mathbf{y}}_1,\cdots,\bar{\mathbf{y}}_N,\mathbf{r}_{t+1})|\mathbf{r}_t]$ is the conditional expectation on the prices of the secondary markets in period $t$, $\mathbf{r}_t$. The echelon inventory positions are bounded by $0\leq y_1^0\leq\cdots\leq y_1^{L_1-1}\leq y_2^0\leq\cdots\leq y_N^{L_N-1}$ and formed into a sublattice. In addition, the constraints on the echelon inventory positions and decision variables enable the inventory positions in the next period to fulfill $\bar{y}_1^0\leq\cdots\leq\bar{y}_1^{L_1-1}\leq\bar{y}_2^0\leq\cdots\leq\bar{y}_N^{L_N-1}$ and to form a sublattice. To demonstrate that $f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)$ is a multimodular function, it suffices to demonstrate that for given prices $\mathbf{r}_t$,

$$\begin{aligned}&\Psi_t(\eta,\mathbf{u}_1,\cdots,\mathbf{u}_N,\mathbf{r}_t)\\ =&\min_{\substack{y_{i+1}^0\geq y_i^{L_i}+z_{i+1},y_i^{L_i}\geq y_i^{L_i-1},\\ z_{i+1}\geq 0,i\in\{1,\cdots,N-1\},y_N^{L_N}\geq y_N^{L_N-1}}}\left\{\begin{array}{c}\sum_{i=1}^N c_i(u_i^{L_i}-u_i^{L_i-1})-\sum_{i=2}^N(r_i+\sum_{k=i}^N h_k)z_i\\ +\sum_{i=1}^N h_i(u_i^0-ED)+(b+\sum_{i=1}^N h_i)E(u_1^0-D)^-\\ +\beta E[\Psi_{t+1}(\eta,\bar{\mathbf{u}}_1,\cdots,\bar{\mathbf{u}}_N,\mathbf{r}_{t+1})|\mathbf{r}_t]\end{array}\right\}\end{aligned} \tag{4.5}$$

is submodular in $(\mathbf{u}_1,\cdots,\mathbf{u}_N)$ and a real $\eta$, where $u_i^j=y_i^j+\eta$ $(i=1,\cdots,N,j=0,1,\cdots,L_i-1)$.

We will adopt a preliminary result to preserve decomposability for optimal cost function out of optimizing selling decisions at all the secondary markets. The result is written in ensuing lemma but its proof is put in Appendix for the reader's reference.

LEMMA 1. *Suppose that $H(y_0,\cdots,y_l)$ is a submodular cost function in a convex sublattice of $\mathbb{R}^{l+1}$ subject to $y_0\leq y_1\leq\cdots\leq y_l$. $\bar{H}(y_0,y_1,\cdots,y_l)$ is the optimal cost function out of the following decision-making: for a given $0\leq k\leq l$,*

$$\bar{H}(y_0,y_1,\cdots,y_l)=\min_{0\leq q\leq y_k-y_{k-1},k=1,\cdots,l}\{\alpha q+H(y_0,y_1,\cdots,y_k-q,\cdots,y_l-q)\} \tag{4.6}$$

*subject to $y_0\leq\cdots\leq y_l$. Then, $\bar{H}(y_0,\cdots,y_l)$ is submodular in $\boldsymbol{y}$ when $y_0\leq\cdots\leq y_l$.*

Clearly, a sum of convex functions in the echelon inventory positions is multimodular in the stage variables, and so is a decomposable terminal function.

By Theorem 1, in period $t$ $(t=1,\cdots,T)$, optimal cost function $f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)$ is multimodular in the stage variables and submodular in the echelon inventory positions. Thus, the echelon inventory positions are complementary in line with $f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)$.

The ensuing lemma shows that the multimodularity can be preserved in period $t$ by optimal order, shipment and sales decisions.

LEMMA 2. *Suppose that in period $t$, for given prices $\boldsymbol{r}_t$ in the secondary markets, optimal cost function $f_t(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r}_t)$ is multimodular, or is submodular in terms of the echelon inventory positions, i.e., $\Psi_t(\eta,\boldsymbol{u}_1,\cdots,\boldsymbol{u}_N,\boldsymbol{r}_t)$ is submodular in $(\eta,\boldsymbol{u}_1,\cdots,\boldsymbol{u}_N)$, where $u_i^j=y_i^j+\eta$ $(i=1,\cdots,N,j=0,1,\cdots,L_i-1)$.*

We apply Lemma 2 to entail that when the terminal cost function is decomposable, optima cost functions are multimodular.

THEOREM 1. *Suppose that terminal function $f_{T+1}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r}_{T+1})$ is multimodular in $(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N)$ for prices $\boldsymbol{r}_{T+1}$. Then, in period $t$, ($t=1,\cdots,T$), optimal cost function $f_t(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r}_t)$ is also multimodular.*

**Proof:** We prove the current theorem by backward induction. By assuming that the optimal cost function of period $t+1$, $f_{t+1}(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_{t+1})$, is multimodular, as suggested by Lemma 2, $f_t(\cdot,\mathbf{r}_{t+1})$ is multimodular as well. Therefore, optimal cost function $f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)$ is multimodular. Q.E.D.

We point out that the multimodularity in the classic serial multi-echelon inventory model with lost-sales has already been known preservable through optimal ordering/shipment decisions in Tim Huh and Janakiraman (2008) in its equivalent $L^\natural$ convexity. Also, as they showed, the $L^\natural$-convexity or multimodularity enables the *analysis of monotone comparative statics* to derive bounded sensitivities on optimal order/shipment quantities. We extend this result to a serial multi-echelon inventory system with secondary markets in the lost-sales context.

### 4.3. Decomposability Preservation

We verify that optimal order, shipment and sales policies preserve decomposability for optimal cost functions so that optimal cost functions are the sums of echelon inventory positions for this serial multiechelon inventory model with secondary markets. To go forward, we assume that the terminal function $f_{T+1}(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r})=v_T(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r})$ is decomposable as a sum of convex functions in the echelon inventory positions as in (4.3),

To show, by induction, that $f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_t)$ is decomposable for given prices $\mathbf{r}_t$ in period $t$ ($t=1,\cdots,T$), we assume that optimal cost function $f_{k+1}(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}_{t+1})$ ($k=t,\cdots,T$) is decomposable. In reference to Corollaries 1 and 2 of Lemma 1, we will ensure that optimal cost functions are decomposable for the current serial multiechelon inventory model with secondary markets.

COROLLARY 1. *Suppose that $H(y_0,\cdots,y_l)=H_1(y_0,\cdots,y_i)+H_2(y_{i+1},\cdots,y_l)$, $H_1(y_0,\cdots,y_i)$ is a submodular function subject to $y_0\le y_1\le\cdots\le y_i$ and $H_2(y_{i+1}-\eta,\cdots,y_l-\eta)$ is submodular in $(y_{i+1},\cdots,y_l)$ and $\eta$ subject to $y_{i+1}\le\cdots\le y_l$. Then, $\bar{H}(y_0,\cdots,y_l)=\bar{H}_1(y_0,\cdots,y_i)+\bar{H}_2(y_{i+1},\cdots,y_l)$ where $\bar{H}_1(y_0,\cdots,y_i)$ is submodular subject to $y_0\le\cdots\le y_i$ and $\bar{H}_2(y_{i+1}-\eta,\cdots,y_l-\eta)$ is submodular in $(y_{i+1},\cdots,y_l)$ and $\eta$ subject to $y_{i+1}\le\cdots\le y_l$.*

Following Corollary 1, the next corollary is critical to ensure decomposability for optimal cost functions.

Corollary 2. *Suppose that $H(y_0,\cdots,y_l)=\sum_{j=0}^{l}H_j(y_j)$ and for $i=0,1,\cdots,l$, $H_i(y_i)$ is a convex function. Then, $\bar{H}(y_0,\cdots,y_l)=\sum_{j=0}^{l}\bar{H}_j(y_j)$ where for $j=0,1,\cdots,l$, $\bar{H}_j(x)$ is a convex function as well.*

The ensuing lemma assures that decomposable structure can be preserved for $f_t(\cdot,\mathbf{r}_t)$ by optimal order/shipment and sales decisions in period $t$ from $f_{t+1}(\cdot,\mathbf{r}_{t+1})$.

Lemma 3. *Suppose that $f_{t+1}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r}_{t+1})$ is decomposable in the echelon inventory positions, i.e., $f_{t+1}(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r}_{t+1})=\Psi_{t+1}(0,\boldsymbol{y}_1,\cdots,\boldsymbol{y}_N,\boldsymbol{r}_{t+1})=\sum_{i=1}^{N}\sum_{j=0}^{L_i-1}\psi_{ij,t+1}(y_i^j,\boldsymbol{r}_{t+1})$. Then, optimal cost function $f_t(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r}_t)$ is a sum of convex cost functions in the echelon inventory positions:*

$$\Psi_t(0,\boldsymbol{y}_1,\cdots,\boldsymbol{y}_N,\boldsymbol{r}_t)=\sum_{i=1}^{N}\sum_{j=0}^{L_i-1}\psi_{ij,t}(y_i^j,\boldsymbol{r}_t)$$

*where $\psi_{ij,t}(\cdot,\boldsymbol{r})$ is convex for any $\boldsymbol{r}\in\mathbb{R}_+^N$ $(i=1,\cdots,N,j=0,\cdots,L_i-1)$.*

By Lemma 3, we will guarantee in ensuing theorem that the decomposability can be relayed from a decomposable terminal cost function to all the optimal cost functions.

Theorem 2. *Suppose that for any price vector $\boldsymbol{r}$, terminal cost function $v_T(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r})$ is decomposable in the echelon inventory positions such as $v_T(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r})=\sum_{i=1}^{N}\sum_{j=0}^{L_i-1}\psi_{ij,T}(y_i^j,\boldsymbol{r})$. Then, for $t=1,\cdots,T$, optimal cost function $f_t(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r})$ is decomposable in the echelon inventory positions such that for any given prices $\boldsymbol{r}$, and $i=1,\cdots,N,j=0,\cdots,L_i-1$, there exists a convex function $\psi_{ij,T}(y_i^j,\boldsymbol{r})$ of $y_i^j$, such that $f_t(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N,\boldsymbol{r})=\sum_{i=1}^{N}\sum_{j=0}^{L_i-1}\psi_{ij,t}(y_i^j,\boldsymbol{r})$.*

### 4.4. Nested Optimal Order-up-to and Sell-down-to Policies

In this subsection, we lighten optimal order/shipment and sales decision rules at all the stages in each period by assuming optimal cost functions are decomposable as those in (4.2). In period $t$, we denote by $(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r})$ a state observation at the beginning of period $t$ $(t=1,\cdots,T)$ and assume that optimal cost functions are decomposable such that

$$f_t(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r})=\sum_{i=1}^{N}\sum_{j=0}^{L_i-1}\psi_{ij,t}(y_i^j,\mathbf{r}).$$

For convenience, we denote by $Z_k=\sum_{i=1}^{k}z_i$ for $k=1,\cdots,N$, an aggregation of the secondary-market sales at the current and downstream stages, and $y_i^{L_i}, i=1,\cdots,N$, a ship-up-to level at stage $i$. We are aimed to optimize these *nested* sales and order/shipment decision variables in period $t$ so as to minimize the objective of the echelon formulation

$$-\sum_{i=1}^{N}c_iy_i^{L_i-1}+\sum_{i=1}^{N}h_i(y_i^0-ED)$$

$$
\begin{aligned}
&+\sum_{i=1}^{N} c_i y_i^{L_i} + (b+\sum_{i=1}^{N} h_i)E(y_1^0 - Z_1 - D)^- - \sum_{i=1}^{N}(r_i + \sum_{k=i}^{N} h_k)(Z_i - Z_{i-1})\\
&+\beta\sum_{i=1}^{N}\{\sum_{k=0}^{L_i-2} E[\psi_{ik,t+1}(y_i^{k+1} - Z_i - D, \mathbf{r}_{t+1})|\mathbf{r}_t] + E[\psi_{i,L_i-1,t+1}(y_i^{L_i} - Z_i - D, \mathbf{r}_{t+1})|\mathbf{r}_t]\}\\
&= -\sum_{i=1}^{N} c_i y_i^{L_i-1} + \sum_{i=1}^{N} h_i(y_i^0 - ED) + \{c_1 y_1^{L_1} + (b+\sum_{i=1}^{N} h_i)E(y_1^0 - Z_1 - D)^-\\
&-(r_1 - r_2 + h_1)Z_1 + \sum_{k=0}^{L_1-1} E[\psi_{1k,t+1}(y_1^{k+1} - D - Z_1, \mathbf{r}_{t+1})|\mathbf{r}_t]\}\\
&+\sum_{i=2}^{N-1}\{c_i y_i^{L_i} - (r_i - r_{i+1} + h_i)Z_i + \beta\sum_{k=0}^{L_i-1} E[\psi_{ik,t+1}(y_i^{k+1} - Z_i - D, \mathbf{r}_{t+1})|\mathbf{r}_t]\}\\
&+\{-(r_N + h_N)Z_N + c_N y_N^{L_N} + \beta\sum_{k=0}^{L_N-1} E[\psi_{Nk,t+1}(y_N^{k+1} - Z_N - D, \mathbf{r}_{t+1})|\mathbf{r}_t]\},
\end{aligned} \tag{4.7}
$$

subject to $0 \le Z_1 \le y_1^0, Z_{i-1} \le Z_i \le Z_{i-1} + (y_i^0 - y_{i-1}^{L_{i-1}-1})$ and $y_i^{L_i-1} \le y_i^{L_i} \le y_{i+1}^0 - (Z_{i+1} - Z_i)$ $(i=1,\cdots,N-1)$ where $y_0^{L_0-1}=0$ for convention. Clearly, their optimality is equal to that of the original decision variables.

There are $N$ stage objective functions in (4.7). The $i$-th objective function formally addresses optimal order and sales operations at stage $i$ and can be optimized with a pair of decision variables, $(Z_i, y_i^{L_i})$, $(i=1,\cdots,N)$, where $Z_i$ is the down-sales aggregation cumulating sales at stage $i$ and all its downstream stages, and $y_i^{L_i}$ is the top echelon inventory position on a partial serial inventory system consisting of stages from 1 through $i$, when down sales are not deleted, respectively. These new decision variables are *nested* by nature, and thus, optimal policies on them are called optimal *nested* order-up-to and down-sales policies. However, the optimality of these nested variables are only nominate, no matter whether the objective functions are decomposable.

The stage operations optimization is the core of (Clark and Scarf 1960) to decompose a long multi-stage optimization problem into a series of single-item single-location inventory problems, circumventing the curse of dimensions growing with $N$. Optimal nested decision quantities for the stage problems obviously are entirely dependent on the constraints, which involve the optimization results of other stage optimizaitons. But our question is whether the stage optimization can be utilized to contrive an efficient algorithm.

## 5. Efficient Algorithms for Optimal Policies

From the decomposition analysis of classic series multi-stage inventory system (Clark and Scarf 1960), stage optimizations that are the core of the seminar work come to circumvent the curse of dimensions, subsequently solving single-location inventory problem. According to the approach of Clark and Scarf (1960), auxiliary penalty functions must be evaluated recursively at the stages, and consequently, their valuations rend a burdensome dynamic programming procedure. Unfortunately, as Angelus (2011) points out, because the selling decisions depend on sales in downstream stages, such the approach is not applicable to the current series multiechelon system integrating secondary markets. By virtue of the *complementarity analysis*, we contrive efficient algorithms to compute optimal nested order/ship-up-to and sell-down-to rules respectively over the stages (installations).

### 5.1. Stage Optimizations

We assume that for given prices $\mathbf{r}$ in the secondary markets, $f_{t+1}(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r})$ is a decomposable function of the echelon inventory positions, such as,

$$\Psi_{t+1}(\mathbf{y}_1,\cdots,\mathbf{y}_N,\mathbf{r}) = f_{t+1}(\mathbf{x}_1,\cdots,\mathbf{x}_N,\mathbf{r}) = \sum_{i=1}^{N}\sum_{j=0}^{L_i-1}\psi_{ij,t+1}(y_i^j,\mathbf{r})$$

is a sum of convex functions in the echelon inventory positions $(\mathbf{y}_1,\cdots,\mathbf{y}_N)$.

The stage optimizations aim to minimize costs incurred out of sales and inventory replenishment operations, respectively, at the stages. By rearranging terms in (4.7), the objective function can be expressed concisely into a function of $(\mathbf{z},\mathbf{y})$, such as,

$$v(\mathbf{z},\mathbf{y}) = \sum_{i=1}^{N}[a_i z_i + \alpha_{i1}(z_i) + b_i y_i + \alpha_{i2}(y_i - z_i)],$$

complying with $0 \le z_1 \le z_2 \le \cdots \le z_N, z_1 \le y_1^0$, $y_i - z_i \le y_{i+1}^0 - z_{i+1}, i = 0,1,\cdots,N-1$ and $y_i^{L_i-1} \le y_i, i = 1,\cdots,N$. The $i$-th stage optimization minimize an objective cost function with two decisive variables $(z_i, y_i)$, $a_i z_i + b_i y_i + \alpha_{i1}(z_i) + \alpha_{i2}(y_i - z_i)$, subject to $c_i \le z_i \le d_i, c_i' \le y_i \le d_i'$, where for $i,j = 1,2$, $\alpha_{ij}(\cdot)$, is a convex function and $c_i, c_i', d_i$ and $d_i'$ may contain either of $(z_{i-1}, y_{i-1})$ or $(z_{i+1}, y_{i+1})$. Thus, the constraints on down-sales aggregations $\mathbf{z}$ and order(ship)-up-to inventory positions $\mathbf{y}$ make these stage minimizations correlated. An important mathematical issue should not be ignored: that is, the feasible collection of $(\mathbf{z},\mathbf{y})$ is not a sublattice by the constraints, and hence, the objective function of (4.7) is not submodular in $(\mathbf{z},\mathbf{y})$. In fact, as for $i = 1,\cdots,N$, $y_i - z_i \le y_{i+1}^0 - z_{i+1}$, the collection of $(\mathbf{z},\mathbf{y})$, by Topkis (1998), is not a sublattice, and thus, $v(\mathbf{z},\mathbf{y})$ is not submodular. To make use of the complementarity analysis and to increase computational efficiency, when $\mathbf{z}$ are fixed, we substitute $\mathbf{y} = (y_1,\cdots,y_N)$ for $\bar{\mathbf{z}} = (\bar{z}_1,\cdots,\bar{z}_N)$ such that $\bar{z}_i = z_i - y_i$ for $i = 1,\cdots,N$. As a result, the new decision variables $(\mathbf{z},\bar{\mathbf{z}})$ are collected by, for $i = 1,\cdots,N-1$, $z_{i+1} - y_{i+1}^0 \le \bar{z}_i \le z_i - y_i^{L_i-1}$ and $z_{i+1} - z_i \le y_{i+1}^0 - y_i^{L_i-1}$, into a sublattice.

Consequently, the objective function

$$v(\mathbf{z},\mathbf{y}) = \bar{v}(\mathbf{z},\bar{\mathbf{z}}) = \sum_{i=1}^{N}[(a_i + b_i)z_i + \alpha_{i1}(z_i) - b_i\bar{z}_i + \alpha_{i2}(-\bar{z}_i)],$$

becomes a submodular function of $\bar{v}(\mathbf{z},\bar{\mathbf{z}})$. At stage $i$ $(i=1,2,\cdots,N-1)$, the $i$-th stage optimization minimizes $-b_i\bar{z}_i+\alpha_{i2}(-\bar{z}_i)$ subject to $z_{i+1}-y^0_{i+1}\le\bar{z}_i\le z_i-y_i^{L_i-1}$ when assuming $z_{i+1}-z_i\le y^0_{i+1}-y_i^{L_i-1}$, and at stage $N$, the $N$-th optimization minimizes $-b_N\bar{z}_N+\alpha_{N2}(-\bar{z}_N)$ subject to $\bar{z}_N\le z_N-y_N^{L_N-1}$. They actually solve for stage $i$ $(i=1,\cdots,N-1)$,

$$\theta_i(z_i,z_{i+1})=\min_{z_{i+1}-y^0_{i+1}\le\bar{z}_i\le z_i-y_i^{L_i-1}}[-b_i\bar{z}_i+\alpha_{I2}(-\bar{z}_i)]+(a_i+b_i)z_i+\alpha_{i1}(z_i)$$

and for stage $N$,

$$\theta_N(z_N)=\min_{\bar{z}_N\le z_N-y_N^{L_N-1}}[-b_N\bar{z}_N+\alpha_{N2}(-\bar{z}_N)]+(a_N+b_N)z_N+\alpha_{N1}(z_N),$$

together. It is worthwhile noting that constraints $z_{i+1}-y^0_{i+1}\le\bar{z}_i\le z_i-y_i^{L_i-1}$ $(i=1,\cdots,N-1)$ in preceding stage optimizations imply $z_{i+1}-z_i\le y^0_{i+1}-y_i^{L_i-1}$ $(i=1,\cdots,N-1)$. If at stage $i$, this constraint does not hold, then, $\theta_i(z_i,z_{i+1})$ will not make sense. Thereby, a feasible sequence of down-sales aggregations $\mathbf{z}$ must satisfy $0\le z_1\le y^0_1, z_1\le z_2\le\cdots\le z_N$ and $z_{i+1}-z_i\le y^0_{i+1}-y_i^{L_i-1}$ for $i=1,\cdots,N-1$.

Furthermore, the stage optimizations will minimize $\theta_i(z_i,z_{i+1})$ subject to $z_{i+1}\ge z_i\ge 0$ and $z_{i+1}-z_i\le y^0_{i+1}-y_i^{L_i-1}$ at stage $i$ $(i=1,\cdots,N-1)$, and $\theta_N(z_N)$, respectively. To this end, by denoting by $\bar{z}^*_i=\arg\min_z[-b_iz+\alpha_{i2}(-z)]$ the global minimizer of the $i$-th stage optimization, we express, for $i=1,\cdots,N-1$,

$$\begin{aligned}\theta_i(z_i,z_{i+1}) &= -b_i[\bar{z}^*_i\vee(z_{i+1}-y^0_{i+1})]+\alpha_{i2}(-(\bar{z}^*_i\vee(z_{i+1}-y^0_{i+1})))\\ &\quad -b_i[\bar{z}^*_i\wedge(z_i-y_i^{L_i-1})]+\alpha_{i2}(-(\bar{z}^*_i\wedge(z_i-y_i^{L_i-1})))\\ &\quad +b_i\bar{z}^*_i-\alpha_{i2}(-\bar{z}^*_i)+(a_i+b_i)z_i+\alpha_{i1}(z_i)\end{aligned}$$

into a sum of two convex functions, where $-b_i[\bar{z}^*_i\vee(z_{i+1}-y^0_{i+1})]+\alpha_{i2}(-(\bar{z}^*_i\vee(z_{i+1}-y^0_{i+1})))$ is a decreasing convex function of $z_{i+1}$ and $-b_i[\bar{z}^*_i\wedge(z_i-y_i^{L_i-1})]+\alpha_{i2}(-(\bar{z}^*_i\wedge(z_i-y_i^{L_i-1})))$ is an increasing and convex function of $z_i$. In addition, we express

$$\theta_N(z_N)=-b_N[\bar{z}^*_N\vee(z_N-y_N^{L_N-1})]+\alpha_{N2}(-\bar{z}^*_N\vee(z_N-y_N^{L_N-1}))+(a_N+b_N)z_N+\alpha_{N1}(z_N).$$

Therefore, $\sum_{i=1}^{N-1}\theta_i(z_i,z_{i+1})+\theta_N(z_N)$ is solved by a sum of cost functions in a given sequence of down-sales aggregations $0\le z_1\le\cdots\le z_N$ in the secondary markets. Rearranging terms it is lead to

$$\sum_{i=1}^{N-1}\theta_i(z_i,z_{i+1})+\theta_N(z_N)=\sum_{i=1}^{N}\bar{\theta}_i(z_i)+\sum_{i=1}^{N}[b_i\bar{z}^*_i-\alpha_{i2}(-\bar{z}^*_i)]$$

where for $i=2,\cdots,N$,

$$\begin{aligned}\bar{\theta}_i(z_i) &= -b_i[\bar{z}^*_i\wedge(z_i-y_i^{L_i-1})]+\alpha_{i2}(-(\bar{z}^*_i\wedge(z_i-y_i^{L_i-1})))\\ &\quad -b_{i-1}[\bar{z}^*_{i-1}\vee(z_i-y^0_i)]+\alpha_{i-1,2}(-(\bar{z}^*_{i-1}\vee(z_i-y^0_i)))+(a_i+b_i)z_i+\alpha_{i1}(z_i),\end{aligned}$$

and for $i=1$,

$$\bar{\theta}_1(z_1) = -b_1[\bar{z}_1^* \wedge (z_1 - y_1^{L_1-1})] + \alpha_{12}(-(\bar{z}_1^* \wedge (z_1 - y_1^{L_1-1})) + (a_1+b_1)z_1 + \alpha_{11}(z_1).$$

Thereby, the stage optimizations that are simultaneously minimizing $\sum_{i=1}^{N-1}\theta_i(z_i, z_{i+1}) + \theta_N(z_N)$ subject to $0 \le z_1 \le y_1, z_1 \le z_2 \le \cdots \le z_N$ and $z_{i+1} - z_i \le y_{i+1}^0 - y_i^{L_i-1}$ results in that of minimizing $\sum_{i=1}^N \bar{\theta}_i(z_i)$ over the same constraints. When order/shipment decisions at all the stages are optimized, the original stage optimizations have been modified into an equivalence of $N$ independent stage optimizations. The stage optimizations attain optimal down-sales aggregations at stage $i$ by $z_i^{(1)} = \arg\min_{z\ge 0}\bar{\theta}_i(z)$ at stage $i$ $(i=1,\cdots,N)$.

### 5.2. Interstage Optimization

However, the outcome of the stage optimization at stage $i$, $\mathbf{z}^{(1)} = (z_1^{(1)},\cdots,z_N^{(1)})$, may not be optimal and even feasible to the echelon optimization. Fortunately, they will become optimal via the *interstage optimizations* to sort out optimal sales quantities. Two infeasible situations may be encountered by an outcome, $\mathbf{z}^{(1)}$, of the stage optimizations: for some $i=1,\cdots,N-1$, (i) $z_i^{(1)} > z_{i+1}^{(1)} \ge 0$, where $z_{i+1}^{(1)} = 0$ is allowed, and (ii) $z_{i+1}^{(1)} - z_i^{(1)} > y_{i+1}^0 - y_i^{L_i-1}$. To adjust infeasible solutions to modify $\mathbf{z}^{(1)}$ into feasible ones, we call for the interstage balance optimization to minimize $\sum_{i=1}^N \bar{\theta}_i(z_i)$ subject to $z_i + y_{i+1}^0 - y_i^{L_i-1} \ge z_{i+1} \ge z_i \ge 0$ $(i=1,\cdots,N-1)$.

The following lemma will be used in the interstage optimization procedure to correct across-stages infeasibility.

Lemma 4. *Suppose $\pi_i(x_i)$ is a convex function and has finite minimizer $x_i^{(1)} = \arg\min_{x_i}\pi_i(x_i)$ $(i=1,2)$. Then, for real $a$ and $b$ $(a<b)$, and $A>0$, if $0 \le x_2^{(1)} - x_1^{(1)} \le A$, then,*

$$\min_{a\le x_1\le x_2\le b, x_2-x_1\le A}[\pi_1(x_1)+\pi_2(x_2)] = \pi_1((x_1^{(1)}\vee a)\wedge b) + \pi_2((x_2^{(1)}\vee a)\wedge b) \tag{5.8}$$

*and if $x_2^{(1)} - x_1^{(1)} > A$, then,*

$$\min_{a\le x_1\le x_2\le b, x_2-x_1\le A}[\pi_1(x_1)+\pi_2(x_2)] = \pi_1((x_1^{(1)}\vee a)\wedge b) + \pi_2([(x_2^{(1)}\vee a)\wedge b]\wedge[(x_1^{(1)}\vee a)\wedge b + A]). \tag{5.9}$$

*Otherwise, if $x_2^{(1)} - x_1^{(1)} \le 0$, then,*

$$\min_{\substack{a\le x_1\le x_2\le b\\ x_2-x_1\le A}}[\pi_1(x_1)+\pi_2(x_2)] = \pi_1((x_1^{(2)}\vee a)\wedge b) + \pi_2((x_2^{(2)}\vee a)\wedge b) \tag{5.10}$$

*where $x_1^{(2)}$ is a minimizer for*

$$\min_{a\le x\le b}[\pi_1(x)+\pi_2(x)]. \tag{5.11}$$

Note that when $\bar{\theta}'_{i+1}(z_{i+1}) \geq 0$ for $z_{i+1} \geq 0$, we set $z_{i+1}^{(1)} = 0$ conventionally. By Corollary **??**, when $z_i^{(1)} > z_{i+1}^{(1)}$,

$$\min_{z_{i-1} \leq z_i \leq z_{i+1} \leq z_{i+2}^{(1)}} [\bar{\theta}_i(z_i) + \bar{\theta}_{i+1}(z_{i+1})] > \bar{\theta}_i(z_i^{(1)}) + \bar{\theta}_{i+1}(z_{i+1}^{(1)})$$

and there exists a finite $z_i^{(2)} \geq 0$, which is the minimizer of $\bar{\theta}_i(z) + \bar{\theta}_{i+1}(z)$ subject to $0 \leq z$, such that $z_i^{(1)} > z_i^{(2)} > z_{i+1}^{(1)}$ and

$$\min_{0 \leq z_i \leq z_{i+1} \leq z, z_{i+1} - z_i \leq y_{i+1}^0 - y_i^{L_i - 1}} [\bar{\theta}_i(z_i) + \bar{\theta}_{i+1}(z_{i+1})] = \min_{0 \leq z_i \leq z} [\bar{\theta}_i(z) + \bar{\theta}_{i+1}(z)] = \bar{\theta}_i(z_i^{(2)} \wedge z) + \bar{\theta}_{i+1}(z_i^{(2)} \wedge z).$$

Moreover, when $z_{i+1}^{(1)} - z_i^{(1)} > y_{i+1}^0 - y_i^{L_i - 1} \geq 0$, also,

$$\min_{z_{i-1} \leq z_i \leq z_{i+1} \leq z_{i+2}^{(1)}, z_{i+1} - z_i \leq y_{i+1}^0 - y_i^{L_i - 1}} [\bar{\theta}_i(z_i) + \bar{\theta}_{i+1}(z_{i+1})] > \bar{\theta}_i(z_i^{(1)}) + \bar{\theta}_{i+1}(z_{i+1}^{(1)}).$$

By Lemma 4,

$$\begin{aligned}
&\min_{z_{i-1} \leq z_i \leq z_{i+1} \leq z_{i+2}^{(1)}, z_{i+1} - z_i \leq y_{i+1}^0 - y_i^{L_i - 1}} [\bar{\theta}_i(z_i) + \bar{\theta}_{i+1}(z_{i+1})] \\
&= \bar{\theta}_i((z_i^{(1)} \wedge z_{i+2}^{(1)}) + \bar{\theta}_{i+1}([z_{i+1}^{(1)} \wedge z_{i+2}^{(1)}] \wedge [(z_i^{(1)} \wedge z_{i+2}^{(1)}) + y_{i+1}^0 - y_i^{L_i - 1}]) \\
&= \bar{\theta}_i(z_i^{(1)}) + \bar{\theta}_{i+1}(z_{i+1}^{(1)} \wedge (z_i^{(1)} + y_{i+1}^0 - y_i^{L_i - 1})).
\end{aligned}$$

Based on the preceding two adjustment procedure on the infeasible situations, we are performing interstage optimizations for more than two stages that starts at stage $i$ when $z_i^{(1)} > z_{i+1}^{(1)}$ $(1 \leq i < N-1)$ and next involves stage $i+2$ when $z_i^{(2)} > z_{i+2}^{(1)}$. Consequently, the problem of three stages $\min_{0 \leq z_i \leq z_{i+1} \leq z_{i+2}} [\sum_{k=0}^2 \bar{\theta}_{i+k}(z_{i+k})]$ need demonstrate that

$$\min_{0 \leq z_i \leq z_{i+1} \leq z_{i+2}} [\sum_{k=0}^2 \bar{\theta}_{i+k}(z_{i+k})] = \min_{0 \leq z_i \leq z} \sum_{k=0}^2 \bar{\theta}_{i+k}(z_i).$$

Actually, as $z_i^{(1)} > z_{i+1}^{(1)}$, by the previous analysis,

$$\min_{0 \leq z_i \leq z_{i+1} \leq z_{i+2}} \sum_{k=0}^1 \bar{\theta}_{i+k}(z_{i+k}) = \min_{0 \leq z_i \leq z_{i+2}} [\bar{\theta}_i(z_i) + \bar{\theta}_{i+1}(z_i)] = \bar{\theta}_i(z_i^{(2)} \wedge z_{i+2}) + \bar{\theta}_{i+1}(z_i^{(2)} \wedge z_{i+2})$$

and thus, by the same token,

$$\begin{aligned}
\min_{0 \leq z_i \leq z_{i+1} \leq z_{i+2}} [\sum_{k=0}^2 \bar{\theta}_{i+k}(z_{i+k})] &= \min_{0 \leq z_{i+2}} \{ \min_{0 \leq z_i \leq z_{i+1} \leq z_{i+2}} [\bar{\theta}_i(z_i) + \bar{\theta}_{i+1}(z_{i+1})] + \bar{\theta}_{i+2}(z_{i+2}) \} \\
&= \min_{0 \leq z_{i+2}} \{ \min_{0 \leq z_i \leq z_{i+2}} [\bar{\theta}_i(z_i) + \bar{\theta}_{i+1}(z_i)] + \bar{\theta}_{i+2}(z_{i+2}) \} \\
&= \min_{0 \leq z_i \leq z_{i+2}} \{ [\bar{\theta}_i(z_i) + \bar{\theta}_{i+1}(z_i)] + \bar{\theta}_{i+2}(z_{i+2}) \}.
\end{aligned}$$

If $z_{i+2}^{(1)} < z_i^{(2)}$, then, by Corollary, there exists $z_i^{(3)} \geq 0$ ($z_{i+2}^{(1)} \leq z_i^{(3)} < z_i^{(2)}$) such that

$$\min_{0\leq z_i\leq z_{i+1}\leq z_{i+2}\leq z}[\sum_{k=0}^{2}\bar{\theta}_{i+k}(z_{i+k})] = \min_{0\leq z_i\leq z}\sum_{k=0}^{2}\bar{\theta}_{i+k}(z_i) = \sum_{k=0}^{2}\bar{\theta}_{i+k}(z_i^{(3)}\wedge z).$$

The problems risen by $z_i^{(1)} > z_{i+1}^{(1)}$ and $z_i^{(2)} > z_{i+2}^{(1)}$ is caused by the stock oversold at stage $i$ that incurs larger costs on the sales of the secondary markets at stage $i+1$ and $i+2$. The interstage optimization will optimize sales from stage $i$ in the secondary market when stopping sales at stages $i+1$ and $i+2$.

The analysis for the three-stage interstage optimization is a special case of the following lemma.

Lemma 5. *Suppose that there are $n$ convex functions, $\pi_1(x),\cdots,\pi_n(x)$, and $x_i^{(1)} = \arg\min_x \pi_i(x)$ ($i=1,\cdots,n$) is a minimizer of $\pi_i(x)$. Suppose in addition that letting $x^{(j)} = \arg\min_x \sum_{k=1}^{j}\pi_k(x)$, and $x^{(j)} \geq x_{j+1}^{(1)}$ ($j=1,\cdots,n-1$). Then, for $1\leq j\leq n$,*

$$\min_{x_1\leq x_2\leq\cdots\leq x_j\leq y}\sum_{k=1}^{j}\pi_k(x_k) = \min_{0\leq x\leq y}\sum_{k=1}^{j}\pi_k(x) = \pi_j^*(x^{(j)}\wedge y). \tag{5.12}$$

*where $\pi_j^*(y) = \min_{0\leq x_1\leq\cdots\leq x_j\leq y}\sum_{i=1}^{j}\pi_i(x_i)$.*

The proof of Lemma 5 is given in Appendix by induction on the number of convex functions in a similar manner to the previous proof for three functions $\bar{\theta}_i(z_i), \bar{\theta}_{i+1}(z_{i+1})$ and $\bar{\theta}_{i+2}(z_{i+2})$. It, however, has laid the groundworks for the interstage balance optimization.

For more applications, the lemma is stated differently using a general convex function $\pi_k(x_k)$ rather than using $\bar{\theta}_i(z_i)$. In the algorithmic statements, $\pi_k(x)$ could be equal to some stage cost $\bar{\theta}_i(x_i)$, but also, it is likely that $\pi_k(x) = \sum_{l=i_k}^{j_k}\bar{\theta}_l(x)$, a sum of $j_k - i_k$ stage cost functions.

Let us correct second infeasible situation for a sequence of $\hat{\mathbf{z}} = (\hat{z}_1,\cdots,\hat{z}_N)$ subject to $0\leq\hat{z}_1\leq\cdots\leq\hat{z}_N$ and $\hat{z}_1\leq y_1^0$ to satisfy

$$\min_{0\leq z_1\leq\cdots\leq z_N, z_1\leq y_1^0}\sum_{i=1}^{N}\bar{\theta}_i(z_i) = \sum_{i=1}^{N}\bar{\theta}_i(\hat{z}_i)$$

while for some $i=1,\cdots,N-1$, $\hat{z}_{i+1}-\hat{z}_i > y_{i+1}^0 - y_i^{L_i-1}$ is held. We denote by $i_0 = \min\{i\leq N-1: \hat{z}_{i+1}-\hat{z}_i > y_{i+1}^0 - y_i^{L_i-1}\}$ to indicate the lowest stage at which this constraint is not bound. In general, we assume that $\hat{z}_{i_0+1} = \cdots = \hat{z}_{i_0+j}$ for $1\leq j\leq N-i_0$. By Lemma 5,

$$\min_{\substack{0\leq z_1\leq\cdots\leq z_j \\ z_{k+1}-z_k\leq y_{i_0+k+1}^0 - y_{i_0+k}^{L_{i_0+k}-1} \\ k=1,\cdots,j-1}}\sum_{k=1}^{j}\bar{\theta}_{i_0+k}(z_k) = \min_{z\geq 0}\sum_{k=1}^{j}\bar{\theta}_{i_0+k}(z)$$

and thus, as $\hat z_{i_0+1} > \hat z_{i_0} + y^0_{i_0+1} - y^{L_{i_0}-1}_{i_0}$, by Lemma 4,

$$\begin{aligned}&\min_{\substack{0\le z_0\le z_1\le\cdots\le z_j\\ z_{k+1}-z_k\le y^0_{i_0+k+1}-y^{L_{i_0+k}-1}_{i_0+k}\\ k=0,1,\cdots,j-1}} \sum_{k=0}^{j}\bar\theta_{i_0+k}(z_k)\\ =&\min_{0\le z_0\le z_1}[\bar\theta_{i_0}(z_0)+\sum_{k=1}^{j}\bar\theta_{i_0+k}(z_1)] = \theta_{i_0}(\hat z_{i_0}) + \sum_{k=1}^{j}\bar\theta_{i_0+k}(\hat z_{i_0} + y^0_{i_0+1} - y^{L_{i_0}-1}_{i_0}).\end{aligned}$$

Thereby, we replace $\hat z_{i_0+1} = \cdots = \hat z_{i_0+j}$ with $\hat z^{(1)}_{i_0+1} = \cdots = \hat z^{(1)}_{i_0+j} = \hat z_{i_0} + y^0_{i_0+1} - y^{L_{i_0}-1}_{i_0}$ to complete the first adjustment on the second infeasible situation. By the same token, by denoting by $i_1 = \min\{l > i_0 + j : \hat z_{l+1} - \hat z_l > y^0_{l+1} - y^{L_l-1}_l\}$, we can adjust infeasible situation between $\hat z_{i_1}$ and $\hat z_{i_1+1}$. The adjustment procedure is conducted as the same as we did for these two situations until $\hat z_{N-1}$ and $\hat z_N$ have been dealt with.

The analysis of correcting the second infeasible situation is summarized by ensuing lemma.

Lemma 6. *Suppose that* $\boldsymbol{z}^{(0)} = (z^{(0)}_1, \cdots, z^{(0)}_N)$ *minimizes* $\sum_{i=1}^N \bar\theta_i(z_i)$ *subject to* $0 \le z_1 \le \cdots \le z_N, z_1 \le y^0_1$*. Suppose in addition that* $\boldsymbol{z}^{(1)} = (z^{(1)}_1, \cdots, z^{(1)}_N)$ *is the outcome of the correction procedure for the second infeasible situation, Then,* $\boldsymbol{z}^{(1)}$ *minimizes* $\sum_{i=1}^N \bar\theta_i(z_i)$ *subject to* $0 \le z_1 \le \cdots \le z_N, z_1 \le y^0_1$ *and* $z_{i+1} - z_i \le y^0_{i+1} - y^{L_i-1}_i, i = 1, \cdots, N-1$.

The interstage optimizations consist of a procedure on the algorithm that will correct infeasible situations for optimal quantities.

### 5.3. Efficient Algorithms

Based on the preceding stage and interstage optimization procedures, when with constant prices $\mathbf{r}$, optimal cost function $f_{t+1}(\mathbf{x}_1, \cdots, \mathbf{x}_N, \mathbf{r})$ is additively decomposable as a sum of convex functions $\{\psi_{ik,t+1}(y^k_i, \mathbf{r}) : i = 1, \cdots, N, k = 0, \cdots, L_i - 1\}$ in the echelon inventory positions, the algorithms are developed to attain optimal policy in period $t$. With an initial $(\mathbf{z}^{(0)}, \bar{\mathbf{z}}^{(0)})$ ($\mathbf{z}^{(0)} = (0, \cdots, 0)$ and $\bar{\mathbf{z}}^{(0)} = (-y^0_1, \cdots, -y^0_N)$), they will increase $(\mathbf{z}, \bar{\mathbf{z}})$ subsequently until the objective function (4.7) is minimized. The algorithms start stage optimizations to optimize order/shipments and down-sales aggregations in the secondary markets and then, perform interstage optimizations to remove infeasibility from the stage optimal solutions. For simplicity, we use notation in subsections 5.1 and 5.2 in the algorithm statements.

**The Algorithm Statement of the Stage Optimization**

- Step 1: Compute a minimizer $\bar z^*_i$ for $-b_i\bar z_i + \alpha_{i2}(-\bar z_i)$ at stage $i$ for $i = 1, \cdots, N$ by the first-order equation. Go to Step 2.
- Step 2: Compute functions $\{\bar\theta_1(z_1), \cdots, \bar\theta_N(z_N)\}$ respectively and then, by the first-order equation, for $i = 1, \cdots, N$, compute $z^{(1)}_i$ as the zero of $\bar\theta'_i(z_i)$ at stage $i$. End the algorithm.

The outcomes of the stage optimization algorithm, $\mathbf{z}^{(1)} = (z_1^{(1)}, \cdots, z_N^{(1)})$, will be used as the input in the interstage optimization algorithm.

**The Algorithm Statement of the Interstage Optimizations**

- Step 0: Let $\mathcal{I} = \{1, \cdots, N\}, \mathcal{F} = \{\bar{\theta}_i : i \in \mathcal{I}\}$, and $\mathcal{Z} = \{z_i^{(1)} : i \in \mathcal{I}\}$. Set $J(i) = \min\{i < j \le N : z_j^{(1)} \neq z_i^{(1)}\}$ and $I := 1$.
- Step 1: Start at $I$ to check whether $z_{J(I)}^{(1)} > z_I^{(1)}$. If it is, set $I := J(I)$; otherwise, compute $z_I^{(2)}$ by solving $\bar{\theta}_I(z_I^{(2)}) + \bar{\theta}_{J(I)}(z_I^{(2)}) = 0$ and making sure $z_{J(I)}^{(1)} < z_I^{(2)} < z_I^{(1)}$. Update $(z_I^{(1)}, \cdots, z_{J(I)-1}^{(1)}, z_{J(I)}^{(1)})$ by setting $z_I^{(1)} = \cdots = z_{J(I)}^{(1)} = z_I^{(2)}$ in $\mathcal{Z}$, and $\bar{\theta}_I(z) = \bar{\theta}_{I+1}(z) = \cdots = \bar{\theta}_{J(I)}(z) := \bar{\theta}_I(z) + \bar{\theta}_{J(I)}(z)$ respectively in $\mathcal{Z}$ and $\mathcal{F}$, and set $I := J(I)$. If $I \neq N$, go back to Step 1; otherwise, go to Step 2.
- Step 2: For $i = 1, \cdots, N-1$, check whether $z_i^{(1)} > z_{J(i)}^{(1)}$. If for $i = 1, \cdots, N-1$, $z_i^{(1)} \le z_{J(i)}^{(1)}$, go to Step 3; otherwise, reset $I := 1$ and go to Step 1.
- Step 3: For $i = 1, \cdots, N-1$, check whether $z_{i+1}^{(1)} - z_i^{(1)} \le y_{i+1}^0 - y_i^{L_i - 1}$. If for $i = 1, \cdots, N-1$, $z_{i+1}^{(1)} - z_i^{(1)} \le y_{i+1}^0 - y_i^{L_i-1}$, then, go to Step 4. Otherwise, go to Step 4.
- Step 4: Set $I := 1$ and check whether $z_{I+1}^{(1)} - z_I^{(1)} > y_{I+1}^0 - y_I^{L_I - 1}$. If it is, set $z_{I+1}^{(1)} = \cdots = z_{J(I+1}^{(1)} := z_I^{(1)} + y_{I+1}^0 - y_I^{L_I-1}$ to update $\mathcal{Z}$, and reset $I := J(I+1)$. Otherwise, reset $I := J(I+1)$. If $I = N$, go to Step 5; otherwise, go to Step 4.
- Step 5: For $i = 1, \cdots, N-1$, calculate $\bar{z}_i^{(1)} = [\bar{z}_i^* \vee (z_{i+1}^{(1)} - y_{i+1}^0)] \wedge (z_i^{(1)} - y_i^{L_i-1})$, and for $i = N$, calculate $\bar{z}_N^{(1)} = \bar{z}_N^* \wedge (z_N^{(1)} - y_N^{L_N-1})$, respectively. Then, compute $y_i^{(1)} = z_i^{(1)} - \bar{z}_i^{(1)}$ for $i = 1, \cdots, N$. Stop the algorithm with an output $(\mathbf{z}^{(1)}, \mathbf{y}^{(1)}) = (z_1^{(1)}, \cdots, z_N^{(1)}, y_1^{(1)}, \cdots, y_N^{(1)})$.

In the ensuing theorem, we reveal that the outcome of the algorithm, $(\mathbf{z}^{(1)}, \mathbf{y}^{(1)})$, are optimal sales, ordering/shipping decisions at all the stage.

THEOREM 3. *For given prices $\boldsymbol{r}$ of the secondary markets in period $t$ and an additive decomposable optimal cost function $f_{t+1}(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N, \boldsymbol{r})$, suppose that $(z_i^{(1)}, y_i^{(1)})$ $(i = 1, \cdots, N)$ are a pair of the down-sales aggregation and order/ship-up-to inventory position at stage $i$ that are attained by the stage and interstage algorithms subsequently. Suppose that there are $j$ groups of indices such that the $k$-th group $(1 \le k \le j)$ includes $n_k$ consecutive indices, $i_k, \cdots, i_k + n_k - 1$, and $i_k + n_k - 1 < i_{k+1}$ for $k = 1, \cdots, j-1$ and $i_j + n_j \le N$. In addition, suppose that at group $k$ $(k = 1, \cdots, j)$, $z_{i_k+l+1}^{(1)} = z_{i_k+l}^{(1)}$ for $l = 1, \cdots, n_k - 1$ and for $1 \le i < i_1, i_k + n_k \le i < i_{k+1}, k = 1, \cdots, j$ $(i_{j+1} = N)$, $z_i^{(1)} \le z_{i+1}^{(1)} \le z_i + y_{i+1}^0 - y_i^{L_i - 1}$. Then, for group $k$, $(k = 1, \cdots, j)$*

$$\min_{0 \le z_{i_k} \le \cdots \le z_{i_k + n_k - 1}} \sum_{j=i_k}^{i_k+n_k-1} \bar{\theta}_j(z_j) = \min_{0 \le z} \sum_{j=i_k}^{i_k+n_k-1} \bar{\theta}_j(z)$$

*and $(\boldsymbol{z}.\boldsymbol{y})$ are optimal down-sales aggregations in the secondary markets and order/ship-up-to inventory positions in period $t$.*

The proof of Theorem 3 is in Appendix. Based the theorem, it is confirmed that the algorithms of stage and interstage optimizations produce optimal down-sales aggregations in the secondary markets and order-up-to echelon inventory positions.

## 6. Concluding Remarks

In this paper, we explore a special and important extension to the classic series multiechelon inventory system noticeable from seminar paper Clark and Scarf (1962), as a serial multi-stage inventory/distribution model with the secondary markets, which is initially studied by Angelus (2011). With the advance of electronic commerce, and from energy transportation pipelines, serial supply chains not only store at and transfer finished goods from upstream installations but also sell extra stocks locally as well as remotely from most of the installations. As Angelus (2011) claims, optimal order/shipment and secondary-market sales actions are by nature *nested* order-up-to echelon inventory positions and down-sales aggregations. To determine these nested optimal quantities, we demonstrate that optimal cost functions are additive decomposable in the echelon inventory positions and propose an efficient algorithm to compute them.

Our approach is applicable to deal with serial multi-stage supply chains whose stocks in the upstream stages are allowed to satisfy demands occurring at the upstream stages whenever effectively.

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