# Estimation and Comparison of Beta-Pricing Models*

Siddhartha Chib Guanhao Feng Jingyu He Qianshu Zhang

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**Abstract**

We develop a Bayesian framework for estimating beta-pricing models with traded and nontraded factors, incorporating theory-implied pricing restrictions. The framework enables model comparisons via marginal likelihoods, balancing in-sample fit against model complexity and penalizing specifications that include weak factors. Simulations confirm that this criterion avoids the overfitting bias of traditional metrics such as cross-sectional $R^2$ and Hansen-Jagannathan distance. Our comparison approach provides new insights into how marginal likelihood evaluates factor relevance. A factor with a zero risk premium can still improve model fit if it enters with large loadings that help explain return dynamics. Conversely, a factor with a nonzero risk premium may fail to improve model fit if its contribution to expected returns is insufficient to justify the added complexity. Empirically, we evaluate each nontraded factor for its incremental contribution beyond benchmark traded factor models and find all tested nontraded factors weak. Revisiting Kan, Robotti, and Shanken (2013, *JF*) and Kleibergen and Zhan (2020, *JF*), we find the most supported model excludes all nontraded factors, and all consumption measures are weak when controlling for the market factor.

**Key Words:** Asset Pricing Constraint, Marginal Likelihood, Model Comparison, Nontraded Factors, Risk Premia.

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## 1 Introduction

Beta-pricing models explain cross-sectional differences in expected returns through the exposure of assets to a set of underlying risk factors. Classical examples include the Capital Asset Pricing Model (CAPM) and the multifactor models of Breeden et al. (1989) and Fama and French (1993). These models have provided a theoretical foundation that has guided advances in risk factor identification, risk premia estimation, and empirical testing. Despite these achievements, identifying the relevant factors and estimating their associated premia remain difficult tasks, particularly when weak factors, such as nontraded factors, are present (e.g., Kleibergen, 2009; Gospodinov et al., 2019; Giglio et al., 2025). Two-step estimation procedures that first estimate factor loadings from time-series regressions and then recover risk premia from the cross-section are especially vulnerable to weak factors. In addition, traditional methods for evaluating beta-pricing models, such as cross-sectional $R^2$ and the Hansen–Jagannathan distance, suffer from an overfitting bias. These limitations motivate the need for a unified framework that estimates factor loadings and risk premia in one step, permits consistent model comparisons, and is robust to weak factors.

This paper aims to develop a unified estimation and model comparison framework for the general class of beta-pricing models that contain traded and nontraded factors. Our approach builds on Pastor and Stambaugh (2003) (equations 14–18), where the cross-sectional restrictions, along with the restriction that the risk premia of traded factors equal the mean, are embedded directly into the time-series (TS) model of asset returns. This embedding yields a restricted TS specification in which the factor loadings and risk premia can be estimated in one step, ensuring that the model estimates capture both the dynamic time-series behavior of returns and their cross-sectional structure within a single coherent framework.

To compare beta-pricing models defined by different subsets of traded (T) and nontraded (NT) factors, we supplement the restricted TS model with an auxiliary model that governs the joint evolution of all available factors. This extension ensures that the left-hand side of the TS regressions, asset returns, and factor realizations, remains identical across models, enabling a fair comparison of specifications that differ only in their pricing kernels. The likelihood function for each model then derives from a coherent hierarchical generative process.

In our approach, we compare model variants using Bayesian marginal likelihoods. The key advantage of this method is that it provides a principled trade-off between fit and complexity across different sets of traded (T) and nontraded (NT) factors. Weak or spurious risk sources, which generate nearly constant or near-zero columns in the loading matrix and lead to overparameterization, are automatically penalized through the marginal likelihood. In addition, our framework offers new insights into how factor relevance is evaluated. A factor with a zero risk premium can still enhance model fit if it enters with large loadings that improve the explanation of return dynamics. Conversely, a factor with a nonzero risk premium may fail to improve model fit if its contribution to expected returns is too weak to justify the additional complexity. Thus, unlike traditional approaches that assess factor relevance solely through cross-sectional pricing performance or risk premia significance, our marginal likelihood-based rankings account for *both* cross-sectional and time-series dynamics *and* model complexity. In contrast, we show that criteria such as $\text{CSR}^2$ (which measures the ability of the model to explain variation in average returns) and HJD (which quantifies pricing errors from moment conditions), implemented through the tests of Kan et al. (2013) and Kan and Robotti (2009), tend to overfit.

A key motivation for our focus on model comparisons is the challenge posed by weak factors. This can lead to unreliable risk premium estimates under standard methods such as two-pass regression, maximum likelihood estimation (MLE), and generalized method of moments (GMM). Prior studies have proposed statistical tools that are designed to detect weak factors, including $F$-tests that assess rank conditions (Kleibergen and Zhan, 2020), as well as methods such as test asset selection (Giglio et al., 2025). Rather than testing individual factors for significance, our model comparison framework evaluates overall model fit in light of explanatory power and parsimony. In our framework, models that contain such factors are automatically penalized by the ML and screened out, effectively regularizing the estimation of risk premia. Because of the concern over weak factors, we, therefore, stress model selection over Bayesian model averaging (e.g., Avramov et al., 2023), favoring the identification of models that offer better overall fit by accounting for both in-sample performance and model complexity.[1]

Simulation confirms that our method can balance the model fitness and complexity by excluding weak factors. We also conduct simulation exercises that evaluate risk premia estimation in a correctly specified model and assess model comparison across nearly 1,000 specifications. Under correct specification, our Bayesian risk premia estimates exhibit bias and root mean square error (RMSE) comparable to those from the two-pass weighted least squares (WLS) method. More importantly, the model that maximizes the ML consistently identifies the true model in simulations that use empirically calibrated parameters. In contrast, CSR$^2$ and HJD fail to distinguish the true model from weaker alternatives, highlighting the limitations of conventional metrics that do not account for model complexity and fit in a unified framework.

Our empirical analysis considers monthly U.S. equity data from 1985 to 2023 using a broad cross section of 302 equity portfolios that span industries and multiple characteristics. We examine ten traded factors and eight nontraded factors. The traded factors include the Fama-French five factors, augmented by UMD, BAB, QMJ, IA, and ROE—factors that capture momentum, betting-against-beta, quality, investment, and profitability-related effects. The nontraded factors include industrial production growth, equity market liquidity, long-term interest rates, M2 growth, durable and nondurable consumption growth, the intermediary capital ratio, and presidential approval ratings—variables that reflect macroeconomic and intermediary conditions not directly traded in markets.

First, we assess each nontraded factor's incremental contribution beyond benchmark traded-factor models using marginal likelihood (ML) and find all tested non-

[1] In our empirical studies, however, model selection and averaging are practically equivalent, as the posterior probability of the top-selected model remains close to one.

traded factors to be weak. For example, while LTY and PCEDG slightly improve the CAPM, their effects vanish against stronger benchmarks. Similarly, HKMcapital modestly enhances the Fama-French five-factor (FF5) model but is insignificant relative to the all-traded-factor benchmark. Beyond these, no nontraded factors improve any traded-factor benchmarks. The ML underperformance stems from weak beta estimates, with over two-thirds statistically indistinguishable from zero.

Second, out of all possible combinations, the optimal model includes only eight traded factors, excluding all nontraded ones. The estimated risk premia of SMB, HML, and UMD are insignificant within this eight-factor framework. However, they do account for the time series. Despite the traded factor constraint, IA and ROE are excluded, highlighting their limited marginal contribution within the beta-pricing framework. The second-best model integrates ROE into the rank 1 model but shows posterior betas insignificantly different from zero for two-thirds of test assets, suggesting multicollinearity concerns. Consequently, it is excluded in favor of the rank 1 model. The third-best model incorporates a single nontraded factor, HKMcapital, highlighting the weakness of most nontraded factors.

Third, we assess model performance using the out-of-sample tangency portfolio of test assets. Leveraging the top-performing model's implied mean and covariance estimates for portfolio weights, the resulting tangency portfolio achieves superior out-of-sample performance, with a higher Sharpe ratio and lower downside risk than all other benchmark methods. These results highlight the economic significance of model selection and estimation within our Bayesian framework.

Finally, although Kan et al. (2013) find the ICAPM as the best model (but does not significantly dominate) among those evaluated, we find the Fama-French three-factor (FF3) model delivers the highest ML. Extending the factor set to all 14 factors, the FF3 remains optimal, with no significant contribution from additional nontraded factors. Reevaluating the consumption-based model of Kleibergen and Zhan (2020), we find that controlling for the market factor substantially weakens all consumption measures.

**Related literature.** Our study contributes to the extensive literature that evaluates the statistical performance of factor models. Gibbons, Ross, and Shanken (1989, henceforth GRS test) develops a framework that links the Sharpe ratio of the factor-efficient portfolio to cross-sectional pricing errors. More recently, Fama and French (2018) proposed a maximum squared Sharpe ratio metric that serves as a measure of overall model performance, while Barillas et al. (2020) examined differences in squared Sharpe ratios that help discriminate between competing models. Kan et al. (2024) extends this line of work by introducing methods that account for estimation risk in Sharpe ratio comparisons. Furthermore, Lewellen et al. (2010) critique the reliance on $CSR^2$ and propose refinements to empirical tests to improve inference and robustness. Key suggestions include expanding the set of test assets, imposing theory-consistent constraints, adopting GLS over OLS, and reporting confidence intervals that reflect estimation uncertainty. Our approach incorporates these recommendations by adopting a Bayesian model comparison framework that improves upon traditional beta-pricing tests by evaluating models based on ML, which jointly account for fit and complexity and help guard against overparameterization.

There is also considerable literature that develops Bayesian methods for comparing factor models, primarily in the context of traded factor structures. This includes studies that evaluate model fit and risk premia through Bayesian frameworks, such as Avramov and Chao (2006), Barillas and Shanken (2018), Chib, Zeng, and Zhao (2020), Chib and Zeng (2020), Avramov, Cheng, Metzker, and Voigt (2023), and Chib, Zhao, and Zhou (2024). The approach of Allena (2020) addresses nontraded factors but relies on mimicking portfolios that introduce approximation error into the analysis. Recently, Cong et al. (2023), Feng et al. (2024), and Bie et al. (2024) propose panel tree methods that maximize marginal likelihood, enabling heterogeneous model selection across cross-sectional and time-series dimensions.

Finally, this paper advances the literature on estimating factor risk premia. Two-pass regression methods yield estimates that are prone to bias when factors are weak (e.g., Kan and Zhang, 1999; Kleibergen, 2009). To address this issue, Gospodinov et al.

(2019) develop a maximum likelihood estimation approach that accommodates beta-pricing models under potential misspecification. Recently, Giglio et al. (2025) proposed supervised principal component analysis to recover risk premia associated with weak factors. While these methods represent significant advancements, they lack a unified model selection and estimation framework. This paper addresses this by introducing a systematic approach to evaluate a broad set of models, identifying the ones most consistent with the data. An econometric alternative is the sparse fused GMM developed by Cui et al. (2024), which estimates the conditional SDF moment condition through a structure that promotes time-varying sparse factor models.

The paper is organized as follows. Section 2 details the methodology. Section 3 presents simulation results. Section 4 highlights key empirical findings. In Section 5, we revisit the model comparison in Kan et al. (2013) and the consumption-based model analysis in Kleibergen and Zhan (2020). Finally, Section 6 concludes.

## 2 Methodology

### 2.1 Model Set Up

This section describes the joint time series and cross-section model for the vector of excess returns $\boldsymbol{r}_t : n \times 1$. We start from the stochastic discount factor (SDF) framework in Chib and Zeng (2020), and extend it to include both traded and non-traded factors, where we use superscript $^T$ to denote traded factors (the market and zero-investment returns) and $^{NT}$ for nontraded factors (innovations in macroeconomic variables). We assume that the distributions for factors and returns are normal and take the stationary form

$$\begin{pmatrix} \boldsymbol{f}_t^T \\ \boldsymbol{f}_t^{NT} \\ \boldsymbol{r}_t \end{pmatrix} \sim \mathcal{N}\left( \begin{pmatrix} \boldsymbol{\mu}^T \\ \boldsymbol{\mu}^{NT} \\ \boldsymbol{\mu}_r \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_T & \boldsymbol{\Sigma}_{T,NT} & \boldsymbol{\Sigma}_{T,r} \\ \boldsymbol{\Sigma}_{NT,T} & \boldsymbol{\Sigma}_{NT} & \boldsymbol{\Sigma}_{NT,r} \\ \boldsymbol{\Sigma}_{r,T} & \boldsymbol{\Sigma}_{r,NT} & \boldsymbol{\Sigma}_r \end{pmatrix} \right)$$

This implies that the marginal distribution of factors is

$$\boldsymbol{f}_t \sim \mathcal{N}\left(\boldsymbol{\mu}_f, \boldsymbol{\Sigma}_f\right)$$

where

$$\boldsymbol{f}_t = \begin{pmatrix} \boldsymbol{f}_t^T \\ \boldsymbol{f}_t^{NT} \end{pmatrix}, \quad \boldsymbol{\mu}_f = \begin{pmatrix} \boldsymbol{\mu}^T \\ \boldsymbol{\mu}^{NT} \end{pmatrix}, \quad \boldsymbol{\Sigma}_f = \begin{pmatrix} \boldsymbol{\Sigma}_T & \boldsymbol{\Sigma}_{T,NT} \\ \boldsymbol{\Sigma}_{NT,T} & \boldsymbol{\Sigma}_{NT} \end{pmatrix}$$

and that the return distribution conditional on the factors is given by

$$\boldsymbol{r}_t | \boldsymbol{f}_t^T, \boldsymbol{f}_t^{NT} \sim \mathcal{N}\left(\boldsymbol{\mu}_r + \boldsymbol{\Sigma}_{r,f}\boldsymbol{\Sigma}_f^{-1}(\boldsymbol{f}_t - \boldsymbol{\mu}_f), \boldsymbol{\Sigma}_r - \boldsymbol{\Sigma}_{r,f}\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\Sigma}_{f,r}\right) \tag{1}$$

where $\boldsymbol{\Sigma}_{r,f} = (\boldsymbol{\Sigma}_{r,T} \quad \boldsymbol{\Sigma}_{r,NT})$. Now assuming that these factors are in the SDF $M_t$, we suppose, following Hansen and Jagannathan (1997), that $M_t$ is given by

$$M_t = 1 - \boldsymbol{\lambda}'\boldsymbol{\Sigma}_f^{-1}(\boldsymbol{f}_t - \boldsymbol{\mu}_f) \tag{2}$$

where $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^T, \boldsymbol{\lambda}^{NT})$. Under the no-arbitrage condition, the traded factors and the returns should be priced by the SDF, which generates the pricing restrictions:

$$\mathbb{E}[M_t \boldsymbol{f}_t^{T'}] = 0, \quad \mathbb{E}[M_t \boldsymbol{r}_t'] = 0$$

From the first of these pricing restrictions, we can derive the restriction that

$$\boldsymbol{\mu}^T = \boldsymbol{\lambda}^T$$

The algebra for showing this is straightforward:

$$\begin{aligned}
&\mathbb{E}\left[\left(1 - \boldsymbol{\lambda}'\boldsymbol{\Sigma}_f^{-1}(\boldsymbol{f}_t - \boldsymbol{\mu}_f)\right)\boldsymbol{f}_t^{T'}\right] \\
=&\mathbb{E}\left[\left(1 - \boldsymbol{\lambda}'\boldsymbol{\Sigma}_f^{-1}\begin{pmatrix} \boldsymbol{f}_t^T - \boldsymbol{\mu}^T \\ \boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT} \end{pmatrix}\right)\boldsymbol{f}_t^{T'}\right] \\
=&\boldsymbol{\mu}^{T\prime} - \boldsymbol{\lambda}'\boldsymbol{\Sigma}_f^{-1}\mathbb{E}\left[\begin{pmatrix} \boldsymbol{f}_t^T - \boldsymbol{\mu}^T \\ \boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT} \end{pmatrix}\boldsymbol{f}_t^{T'}\right] \\
=&\boldsymbol{\mu}^{T\prime} - \boldsymbol{\lambda}'\boldsymbol{\Sigma}_f^{-1}\begin{pmatrix} \boldsymbol{\Sigma}_T \\ \boldsymbol{\Sigma}_{NT,T} \end{pmatrix} \\
=&\boldsymbol{\mu}^{T\prime} - \boldsymbol{\lambda}'\begin{pmatrix} \boldsymbol{\Sigma}_T & \boldsymbol{\Sigma}_{T,NT} \\ \boldsymbol{\Sigma}_{NT,T} & \boldsymbol{\Sigma}_{NT} \end{pmatrix}^{-1}\begin{pmatrix} \boldsymbol{\Sigma}_T \\ \boldsymbol{\Sigma}_{NT,T} \end{pmatrix}
\end{aligned}$$

The rest follow by using the formula for the inverse of a block matrix, multiplying it, and setting the expression to zero.

The second pricing condition can be used to show (again by elementary algebra) that

$$\boldsymbol{\mu}_r = \boldsymbol{B}\boldsymbol{\lambda} = \boldsymbol{B}^T\boldsymbol{\lambda}^T + \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} = \boldsymbol{B}^T\boldsymbol{\mu}^T + \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT}$$

where $\boldsymbol{B} = \Sigma_{f,r}\Sigma_f^{-1}$ is the loading matrix. Plugging this restriction into equation 1, we get that

$$\boldsymbol{r}_t|\boldsymbol{f}_t^T, \boldsymbol{f}_t^{NT} \sim \mathcal{N}\left(\boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^T\boldsymbol{f}_t^T + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}), \boldsymbol{\Sigma}_r - \boldsymbol{\Sigma}_{r,f}\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\Sigma}_{f,r}\right)$$

So the return conditional on the factors takes the following restricted linear form:

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^T\boldsymbol{f}_t^T + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t, \boldsymbol{e}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_e). \tag{3}$$

where $\boldsymbol{\Sigma}_e = \boldsymbol{\Sigma}_r - \boldsymbol{\Sigma}_{r,f}\boldsymbol{\Sigma}_f^{-1}\boldsymbol{\Sigma}_{f,r}$. If we also suppose that

$$\boldsymbol{f}_t^{NT} = \boldsymbol{\mu}^{NT} + \boldsymbol{u}_t, \boldsymbol{u}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_u) \tag{4}$$

we have a model in which all parameters can be estimated jointly in a *single step*. Moreover, we can estimate the market prices of factor risks, $\boldsymbol{\Sigma}_u^{-1}\boldsymbol{\lambda}$, from this joint model.

It should be noted that if there is only one NT factor, this restricted beta-pricing model reduces to the formulation in Pastor and Stambaugh (2003).

In practice, one needs an approach to select which factors to include in the beta pricing model. However, the model as written cannot be used for searching over the space of factors. The reason is that as $\boldsymbol{f}_t^{NT}$ changes, the identity of the left-hand side variables changes. This variation implies that the models cannot be compared using a common likelihood scale. To address this issue and enable valid comparisons across competing specifications, we augment the return model with a factor model defined over the full set of factors. Specifically, the model we estimate is given by:

$$\begin{aligned} &\text{RETURN MODEL} \qquad && \boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^T\boldsymbol{f}_t^T + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t \\ &\text{FACTOR MODEL} && \boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t \end{aligned} \tag{5}$$

where $\boldsymbol{f}_t^* = (\boldsymbol{f}_{t1}, \boldsymbol{f}_{t2}) : (k_1^* + k_2^*) \times 1$, $\boldsymbol{f}_{t1}$ is the full set of traded factors, $\boldsymbol{f}_{t2}$ is the full set of nontraded factors, and $\boldsymbol{f}_t^T$ and $\boldsymbol{f}_t^{NT}$ are the subset of traded and nontraded factors, respectively, in the RETURN MODEL, and the first $k^T$ elements of $\boldsymbol{\mu}$ in the FACTOR MODEL equal $\boldsymbol{\lambda}^T$.

This joint model can be viewed as a hierarchical model. The full factor set is generated by the FACTOR MODEL, while returns are modeled by the RETURN MODEL using a subset of $\boldsymbol{f}_t^*$. These models are linked via the parameter $\boldsymbol{\mu}^{NT}$, which corresponds to a subset of the last $k_2^*$ elements of $\boldsymbol{\mu}$. Together, they enable parameter estimation and guide the selection of the most relevant factors from $\boldsymbol{f}_t^*$ to explain returns.

Under our assumption that factors and returns are normally distributed, the innovations follow

$$\boldsymbol{e}_t \sim \mathcal{N}(0, \boldsymbol{\Sigma}_e), \quad \boldsymbol{u}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_u)$$

where, to simplify estimation, we assume that $\boldsymbol{\Sigma}_e = \text{diag}(\sigma_1^2, ..., \sigma_n^2)$ is a diagonal matrix and $\boldsymbol{\Sigma}_u$ is a full covariance matrix. The diagonality assumption facilitates computation in settings with a large number of assets (Avramov and Chordia, 2006). It is also not a binding constraint in practice, as the covariance matrix of residual returns, after accounting for the effects of the factors, tends to be sparse, as illustrated in Figure IA.1. Moreover, simulation studies in which we generate data based on the full error covariance structure observed in the real data, reported in Table IA.4, show that model selection remains unaffected under the diagonal assumption.

**Special cases.** Two special cases can be mentioned. If the model only has NT factors, then our model simplifies to:

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t. \tag{6}$$

which corresponds to Equation (14.209) in Cochrane (2009). When there are only traded factors, we get:

$$\boldsymbol{r}_t = \boldsymbol{B}^T\boldsymbol{f}_t^T + \boldsymbol{e}_t. \tag{7}$$

which corresponds to Equation (14.202) in Cochrane (2009).

### 2.2 Prior Specification

Since we focus on comparing alternative beta pricing models, we impose the condition that the model-specific priors are proper, integrating to one over the parameter space. Additionally, in formulating the priors, we ensure that differences in marginal likelihood reflect genuine model fit rather than variations in prior specifications. Finally, we choose priors that are minimally subjective, requiring little user input while maintaining robustness in inference.

To achieve these objectives, we construct our model-specific prior by utilizing an initial portion of the data, referred to as the training sample, to calibrate the hyperparameters of each prior distribution (Greenberg, 2012). As a default setting, this training sample spans 15% of the total sample period, encompassing $T_0 = 70$ observations in total in our main empirical analysis, denoted as $[\boldsymbol{R}_0, \boldsymbol{F}_0]$, where $\boldsymbol{F}_0 = [\boldsymbol{F}_0^T, \boldsymbol{F}_0^{NT}]$. The training sample is primarily used to determine the mean of the prior distribution, while the dispersion is predominantly governed by training sample information and user-specified hyperparameters that are set with the aim of ensuring sufficient uncertainty regarding the central location. The training sample is excluded from the estimation and model comparison stages to avoid the double use of the data.

Let the parameters of the baseline model be

$$\boldsymbol{\theta} = (\boldsymbol{B}, \boldsymbol{\lambda}^{NT}, \boldsymbol{\Sigma}_e, \boldsymbol{\Sigma}_u, \boldsymbol{\mu}), \quad \boldsymbol{B} = (\boldsymbol{\beta}_1', ..., \boldsymbol{\beta}_n')'. \tag{8}$$

Our prior for $\boldsymbol{\mu}$ is given by:

$$\boldsymbol{\mu} \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{M}_0),$$

where

$$\boldsymbol{\mu}_0 = \bar{\boldsymbol{F}}_0, \quad \boldsymbol{M}_0 = \frac{4}{T_0}(\boldsymbol{F}_0 - \boldsymbol{1}_{T_0}\boldsymbol{\mu}_0')'(\boldsymbol{F}_0 - \boldsymbol{1}_{T_0}\boldsymbol{\mu}_0').$$

The factor of 4 in this expression is a user-specified hyperparameter to widen the distribution around the training sample mean.

For each column of $\boldsymbol{B}$ we suppose that

$$\boldsymbol{\beta}_i \sim \mathcal{N}(\boldsymbol{\beta}_{0i}, \boldsymbol{B}_{0i}),$$

where

$$\boldsymbol{\beta}_{0i} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{r}_{0i}, \quad \boldsymbol{X} = [\boldsymbol{1}_{T_0}, \boldsymbol{F}_0]\ ,$$

$\boldsymbol{r}_{0i}$ is the $i$th column of $\boldsymbol{R}_0$,

$$\boldsymbol{B}_{0i} = 4\sigma_{0i}^2(\boldsymbol{F}_0'\boldsymbol{F}_0)^{-1},$$

and $\sigma_{0i}^2$ is the residual variance in the time series regression of $\boldsymbol{r}_{0i}$ on $\boldsymbol{F}_0$.

Prior distribution for the risk premia of nontraded factors is obtained by taking the prior mean $\boldsymbol{\mu}_0$ and prior mean $\boldsymbol{B}_0$ and fitting the model $\boldsymbol{r}_t - \boldsymbol{B}_0^T \boldsymbol{f}_t^T - \boldsymbol{B}_0^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}_0^{NT}) = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{e}_t$ on the training sample data to give

$$\boldsymbol{\lambda}^{NT} \sim \mathcal{N}(\boldsymbol{\lambda}_0^{NT}, \boldsymbol{L}_0^{NT}).$$

Next, the prior of $\sigma_i$ are

$$\sigma_i \sim \mathcal{IG}\left(\frac{\nu_{e,0i}}{2}, \frac{\delta_{e,0i}}{2}\right),$$

where

$$\nu_{e,0i} = 2(e_i^2/s_i^2 + 2), \quad \delta_{e,0i} = 2e_i(e_i^2/s_i^2 + 1),$$

and $e_i$ is $i^{th}$ residual from the regression $\boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}_0^{NT} + \boldsymbol{B}_0^T \boldsymbol{f}_t^T - \boldsymbol{B}_0^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}_0^{NT}) + \boldsymbol{e}_t$ on the training sample data and $s_i^2$ is the estimated $i$th residual variance from this regression. Finally, the prior for $\boldsymbol{\Sigma}_u$ is

$$\boldsymbol{\Sigma}_u \sim \mathcal{IW}(\rho_{u,0}, \boldsymbol{Q}_{u,0}),$$

where

$$\boldsymbol{Q}_{u,0} = (\boldsymbol{F} - \boldsymbol{1}_{T_0}\boldsymbol{\mu}_0')'(\boldsymbol{F} - \boldsymbol{1}_{T_0}\boldsymbol{\mu}_0'), \quad \rho_{u,0} = T_0.$$

In summary, the training sample prior distribution leverages a sequential Bayesian learning framework to automate the prior specification process. It encodes beliefs about parameter values as of November 1990, preceding the estimation period, and centers the prior distribution accordingly. To account for uncertainty, it inflates the variances and covariances. A key advantage of this approach is its adaptability to the model, ensuring that the priors are consistent and comparable across specifications. Moreover, eliminating the need for manual inputs enables the efficient estimation and comparison of multiple models with minimal effort.

### 2.3 MCMC Simulation Algorithm

We sample the posterior distribution via Markov Chain Monte Carlo (MCMC) methods, within a framework encompassing traded and nontraded factors. The posterior distribution of model parameters, conditional on observed data for $t = 1, \ldots, T$, is proportional to the product of the likelihood function and the prior distribution. The likelihood for the return model is specified as:

$$p(\boldsymbol{r}_t \mid \boldsymbol{f}_t^*, \boldsymbol{B}^T, \boldsymbol{B}^{NT}, \boldsymbol{\lambda}^{NT}, \boldsymbol{\Sigma}_e) = \prod_{t=1}^{T} \mathcal{N}\left(\boldsymbol{r}_t \mid \boldsymbol{B}^T \boldsymbol{f}_t^T + \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}), \boldsymbol{\Sigma}_e\right),$$

where $\boldsymbol{\Sigma}_e = \text{diag}(\sigma_1^2, \ldots, \sigma_n^2)$, and the factor model likelihood is

$$p(\boldsymbol{f}_t^* \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}_u) = \prod_{t=1}^{T} \mathcal{N}(\boldsymbol{f}_t^* \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}_u).$$

The posterior distribution is proportional to the product of the two likelihoods and the prior on the parameters:

$$p(\boldsymbol{B}^T, \boldsymbol{B}^{NT}, \boldsymbol{\lambda}^{NT}, \boldsymbol{\mu}, \boldsymbol{\Sigma}_e, \boldsymbol{\Sigma}_u).$$

We employ MCMC methods to sample from the resulting posterior distribution. Specifically, given the remaining parameters, we sequentially sample from the conditional distributions of parameter blocks to construct a Markov chain whose invariant distribution corresponds to the posterior distribution (Chib, 2001). The conditional distributions for these parameter blocks are derived based on the following formulations of the joint model. For updating $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_e$ we write the joint model as

$$\boldsymbol{r}_t = \boldsymbol{B}\tilde{\boldsymbol{f}}_t + \boldsymbol{e}_t,$$

where

$$\tilde{\boldsymbol{f}}_t = \begin{pmatrix} \boldsymbol{f}_t^T \\ \boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT} + \boldsymbol{\lambda}^{NT} \end{pmatrix}, \tag{9}$$

from which the conditional posterior distributions of $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_e$ can be derived by Bayesian results for multivariate regression models. For updating $\boldsymbol{\lambda}^{NT}$ we rewrite the full model now as

$$\boldsymbol{r}_t - \boldsymbol{B}\boldsymbol{f}_t + \boldsymbol{B}^{NT}\boldsymbol{\mu}^{NT} = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{e}_t.$$

For updating $\boldsymbol{\mu}$ we write the joint model as

$$\boldsymbol{r}_t - \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} - \boldsymbol{B}\boldsymbol{f}_t = -\boldsymbol{B}^{NT}\boldsymbol{\mu}^{NT} + \boldsymbol{e}_t, \tag{10}$$

and

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t,$$

and then derive the conditional posterior of $\boldsymbol{\mu}$ in two steps, first with the return model likelihood and then with the factor model likelihood. Finally, updating $\Sigma_u$ takes place easily using the factor model. The steps in the algorithm are summarized below with full details in Appendix A.I.

**Algorithm** MCMC Sampling with Parameter Expressions

1: **Initialize parameters**: Set $\boldsymbol{\theta}^{(0)} = (\boldsymbol{B}^{(0)}, \Sigma_e^{(0)}, \boldsymbol{\lambda}^{NT(0)}, \Sigma_u^{(0)}, \boldsymbol{\mu}^{(0)})$
2: **for** $g = 1$ to $n_0 + M$ **do**
3: **Step 1:** For $i = 1$ to $i = n$, sample $\boldsymbol{\beta}_i^{(g+1)} \mid \Sigma_e^{(g)}, \boldsymbol{\lambda}^{NT(g)}, \Sigma_u^{(g)}, \boldsymbol{\mu}^{(g)} \sim \mathcal{N}\left(\widehat{\boldsymbol{\beta}}_i^{(g)}, \mathcal{B}_i^{(g)}\right)$
4: **Step 2:** For $i = 1$ to $i = n$, sample $\sigma_{ei}^2 \mid \boldsymbol{B}^{(g+1)}, \boldsymbol{\lambda}^{NT(g)}, \Sigma_u^{(g)}, \boldsymbol{\mu}^{(g)} \sim \mathcal{IG}\left(\nu_{ei}^{(g)}, \delta_{ei}^{(g)}\right)$
5: **Step 3:** Sample $\boldsymbol{\lambda}^{NT} \mid \boldsymbol{B}^{(g+1)}, \Sigma_e^{(g+1)}, \Sigma_u^{(g)}, \boldsymbol{\mu}^{(g)} \sim \mathcal{N}\left(\widehat{\boldsymbol{\lambda}}^{NT(g)}, \boldsymbol{L}^{NT(g)}\right)$
6: **Step 4:** Sample $\boldsymbol{\mu}^{(g+1)} \mid \boldsymbol{B}^{(g+1)}, \Sigma_e^{(g+1)}, \boldsymbol{\lambda}^{NT(g+1)}, \Sigma_u^{(g)} \sim \mathcal{N}\left(\widehat{\boldsymbol{\mu}}^{(g)}, \boldsymbol{D}_1^{(g)}\right)$
7: **Step 5:** Sample $\Sigma_u^{(g+1)} \mid \boldsymbol{B}^{(g+1)}, \Sigma_e^{(g+1)}, \boldsymbol{\lambda}^{NT(g+1)}, \boldsymbol{\mu}^{(g+1)} \sim \mathcal{IW}\left(\nu_u^{(g)}, \boldsymbol{S}_u^{(g)}\right)$
8: **end for**
9: After burn-in, use samples $\{\boldsymbol{\theta}^{(g)}\}_{g=1}^N$ to estimate the posterior distribution.

Thus far, we have developed a model incorporating both traded and nontraded factors. Appendix A.II outlines a special case of this MCMC algorithm for the specification with nontraded factors only (e.g., CCAPM in Breeden et al. (1989)). In Appendix A.III we detail another special case of this MCMC algorithm for the specification with traded factors only (e.g., CAPM).

### 2.4 Marginal Likelihood

Our primary objective in this empirical study is not solely to estimate a single model but also to comprehensively compare multiple competing models. In the Bayesian framework, alternative models are evaluated using posterior odds based on the observed data. The posterior odds provide a quantitative measure of the relative support the data offers for each competing model. Let $p_j = \Pr(\mathcal{M}_j)$ denote the prior probability assigned to model $\mathcal{M}_j$. The posterior odds between model $\mathcal{M}_i$ and $\mathcal{M}_k$ are

$$\frac{\Pr(\mathcal{M}_i \mid \boldsymbol{y}_{1:T})}{\Pr(\mathcal{M}_k \mid \boldsymbol{y}_{1:T})} = \frac{p_i}{p_k}\frac{m(\boldsymbol{y}_{1:T} \mid \mathcal{M}_i)}{m(\boldsymbol{y}_{1:T} \mid \mathcal{M}_k)},$$

where the ML is defined as the integral of the likelihood function with respect to the prior density

$$m(\boldsymbol{y}_{1:T} \mid \mathcal{M}_j) = \int_{\Theta_j} p(\boldsymbol{y}_{1:T} \mid \boldsymbol{\theta}, \mathcal{M}_j)\pi(\boldsymbol{\theta} \mid \mathcal{M}_j)\mathrm{d}\boldsymbol{\theta}.$$

We calculate the ML by the method of Chib (1995).

When prior odds across models are set to one, as assumed here, the posterior odds is equal to the ratio of ML (Bayes Factor), making ML the central criterion for model evaluation. Importantly, the ML criterion does not inherently favor more complex models, mitigating the risk of overfitting. In this context, the role of our training sample priors is crucial. As previously discussed, these priors adapt to each model, ensuring that no model is inadvertently favored — a concern arises when manually specifying priors over a model space comprising thousands of candidate models. Moreover, ML rankings possess strong theoretical properties. As the sample size increases, the model with the highest ML converges to the true model with probability approaching one, or to the closest approximation when all competing models are misspecified (Chib et al., 2018). Further details on the computation of ML are provided in Appendix A.I.

### 2.5 Alternative Estimation Methods

This section reviews several traditional standard methods to estimate the beta-pricing model, but mostly overlooks the traded factor constraint. Let $\boldsymbol{R} = [\boldsymbol{r}_1, \boldsymbol{r}_2, ..., \boldsymbol{r}_T]'$, in the standard two-pass regression, a CS regression is run of the sample mean of return $\bar{\boldsymbol{R}}$ on $\widehat{\boldsymbol{B}}$ to estimate $\boldsymbol{\lambda}$ in the second pass (Kan et al., 2013). Given the weighting matrix $\boldsymbol{W}$, the $\boldsymbol{\lambda}$ can be estimated as:

$$\widehat{\boldsymbol{\lambda}} = (\widehat{\boldsymbol{B}}'\boldsymbol{W}\widehat{\boldsymbol{B}})^{-1}\widehat{\boldsymbol{B}}'\boldsymbol{W}\bar{\boldsymbol{R}}.$$

For OLS, the weighting matrix $\boldsymbol{W}$ is the identity matrix. In GLS, $\boldsymbol{W}$ is set as the inverse of the residual covariance matrix in the initial pass. However, estimating the inverse covariance matrix can be computationally challenging when the number of test assets is large. A practical alternative is the WLS estimator, where $\boldsymbol{W}$ is the inverse of the diagonal elements of the residual covariance matrix.When the test asset count is large in simulations and empirical tests, we opt for WLS over GLS.

Beyond two-pass regression, another approach to estimating the beta-pricing model is MLE (Ang et al., 2020). We combine the TS and CS models into a single likelihood to derive the MLE. The CS regression can be viewed as a restriction on the TS regression. By imposing the pricing restriction in the CS model and taking expectations within the time-series model, then substituting in the TS model, we get the restricted TS model (Cochrane, 2009, Equation 14.209),

$$\boldsymbol{r}_t = \boldsymbol{B}\boldsymbol{\lambda} + \boldsymbol{B}(\boldsymbol{f}_t - E[\boldsymbol{f}_t]) + \boldsymbol{e}_t. \tag{11}$$

For MLE, an auxiliary statistical model in the errors and factors that are i.i.d. normal and uncorrelated with each other, is introduced:

$$\boldsymbol{f}_t = E(\boldsymbol{f}_t) + \boldsymbol{u}_t,$$
$$\begin{bmatrix} \boldsymbol{e}_t \\ \boldsymbol{u}_t \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \boldsymbol{\Sigma} & 0 \\ 0 & \boldsymbol{V} \end{bmatrix}\right).$$

Then, the likelihood function is given by

$$\mathcal{L} \propto -\frac{1}{2}\sum_{t=1}^{T} \boldsymbol{e}_t' \boldsymbol{\Sigma}^{-1} \boldsymbol{e}_t - \frac{1}{2}\sum_{t=1}^{T} \boldsymbol{u}_t' \boldsymbol{V}^{-1} \boldsymbol{u}_t.$$

A more general approach to estimating is GMM, which relaxes the independent and identical distribution assumption of returns. Rewrite TS and CS models into moment conditions (Cochrane, 2009, equation 12.188):

$$\boldsymbol{g}_T = \begin{bmatrix} E\left(\boldsymbol{r}_t - \boldsymbol{a} - \boldsymbol{B}\boldsymbol{f}_t\right) \\ E\left[\left(\boldsymbol{r}_t - \boldsymbol{a} - \boldsymbol{B}\boldsymbol{f}_t\right) \otimes \boldsymbol{f}_t\right] \\ E\left(\boldsymbol{R} - \boldsymbol{B}\boldsymbol{\lambda}\right) \end{bmatrix}. \tag{12}$$

GMM solves the minimization of $\boldsymbol{g}_T \boldsymbol{W} \boldsymbol{g}_T$ for some weighting matrix. An explicit example of GMM estimation under the general model with traded and nontraded factors and traded factor constraint is given in Pastor and Stambaugh (2003) with liquidity as the only nontraded factor.

**Model comparison.** For beta-pricing models, the most commonly used performance measure is CSR$^2$, which is calculated as follows,

$$R^2 = 1 - \frac{\boldsymbol{e}'\boldsymbol{W}\boldsymbol{e}}{\boldsymbol{e}_0'\boldsymbol{W}\boldsymbol{e}_0}, \quad \boldsymbol{e} = \bar{\boldsymbol{R}} - \boldsymbol{B}\widehat{\boldsymbol{\lambda}}, \quad \boldsymbol{e}_0 = [\boldsymbol{I}_N - \mathbf{1}_N(\mathbf{1}_N'\boldsymbol{W}\mathbf{1}_N)^{-1}\mathbf{1}_N'\boldsymbol{W}]\bar{\boldsymbol{R}}$$

where the weighting matrix $\boldsymbol{W}$ depends on the estimation method.

Another widely used method is the HJD (Hansen and Jagannathan, 1997; Kan and Robotti, 2009) metric, computed via the two-stage GMM-SDF framework. Let the covariance matrix between $\boldsymbol{R}$ and $\boldsymbol{F}$ be $\boldsymbol{C}$, and the demeaned factor $\boldsymbol{F}$ be $\tilde{\boldsymbol{F}}$. The SDF loading is defined as $\boldsymbol{b} = (\boldsymbol{C}'\boldsymbol{C})^{-1}\boldsymbol{C}'\bar{\boldsymbol{R}}$. The estimated normalized SDF is $\boldsymbol{m} = 1 - \tilde{\boldsymbol{F}}\widehat{\boldsymbol{b}}$, with residuals $\boldsymbol{R}\boldsymbol{m}$. The squared HJD is $E(\boldsymbol{R}\boldsymbol{m})'E(\boldsymbol{R}\boldsymbol{m})/N^2$, normalized by the number of test assets $N$. However, CSR$^2$ and HJD share a key limitation: larger models weakly dominate smaller ones in fit.

# 3 Simulation

This section assesses the finite-sample performance of our Bayesian estimators via Monte Carlo simulations. We conduct three experiments: Section 3.1 evaluates risk premia estimation under the correctly specified model. Section 3.2 explores nontraded factors under weak identification. Section 3.3 compares model performance across factor sets. All parameters are calibrated using the true data described in Section 4.1.

## 3.1 Simulation under Correctly Specified Model

The primary focus of the beta-pricing model is the estimation of risk premia. Before the model comparison part, we examine whether our model can give comparable estimation results under the correctly specified true model with traditional approaches. We first simulate factors from:

$$\boldsymbol{f}_t = \boldsymbol{\mu} + \boldsymbol{u}_t, \quad \boldsymbol{u}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_u)$$

The $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_u$ are the mean and covariance of the empirical factor data. $\boldsymbol{f}_t = [\boldsymbol{f}^T, \boldsymbol{f}^{NT}]$, where $\boldsymbol{f}^T$ contains 6 commonly used traded factors MktRf, SMB, HML, RMW, CMA, and UMD, while $\boldsymbol{f}^{NT}$ contains two nontraded factors: durable consumption expenditure (PCEDG) and intermediary capital ratio (HKMcapital). We then simulate the asset returns from the following model:

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^T \boldsymbol{f}_t^T + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t, \quad \boldsymbol{e}_t \sim \mathcal{N}(0, \boldsymbol{\Sigma}_e)$$

where $\boldsymbol{r}_t$ contains 100 Fama-French sorted portfolios (first 100 portfolios described in Section 4.1). The factor loadings $\boldsymbol{B} = [\boldsymbol{B}^T \ \boldsymbol{B}^{NT}]$ and the diagonal residual covariance $\boldsymbol{\Sigma}_e$ are calculated from OLS regression of these 100 portfolios on the 8 factors with intercept. The risk premia of nontraded factors $\boldsymbol{\lambda}^{NT}$ is set as $(0.002, 0.003)$, comparable with that of traded factors. The sample period $T$ exactly matches the empirical data, which is 468 months.

Table 1: **Simulation Evidence: Estimation under Correctly Specified Model**

This table presents simulation results for the Correctly Specified model. The analysis incorporates 8 factors —– 6 traded and 2 nontraded. Parameters are calibrated using true test assets and factors. The table reports bias and RMSE for WLS and Bayesian estimation, both with and without the traded factor constraint (NC). Results are based on 1,000 simulations and expressed in percentages.

| | WLS bias | Bayes bias | BayesNC bias | WLS RMSE | Bayes RMSE | BayesNC RMSE |
|---|---|---|---|---|---|---|
| | | | Traded Factors | | | |
| MktRf | -0.0014 | -0.0034 | 0.1684 | 0.2091 | 0.2172 | 0.2772 |
| SMB | 0.0043 | -0.0005 | 0.0384 | 0.1392 | 0.1450 | 0.1503 |
| HML | 0.0065 | -0.0095 | -0.0375 | 0.1432 | 0.1492 | 0.1556 |
| RMW | -0.0101 | -0.0045 | -0.0416 | 0.1193 | 0.1200 | 0.1296 |
| CMA | -0.0001 | 0.0005 | -0.0341 | 0.0999 | 0.0986 | 0.1101 |
| UMD | -0.0289 | 0.0008 | -0.0563 | 0.2309 | 0.2268 | 0.2482 |
| | | | Nontraded Factors | | | |
| PCEDG | -0.0652 | -0.0674 | -0.0525 | 0.2236 | 0.2257 | 0.2264 |
| HKMcapital | 0.0496 | 0.0306 | 0.2158 | 0.3421 | 0.3468 | 0.4170 |

Results from table 1 are based on 1,000 simulations.[2] Table IA.3 in the appendix presents rejection frequencies for the null hypothesis $H_0 : \boldsymbol{\lambda} = \boldsymbol{\lambda}_{true}$. The column "BayesNC" denotes our Bayesian estimation without the traded factor constraint. Although Bayesian estimation uses a training sample and is inherently slightly more biased than WLS, the constrained Bayes estimate achieves bias and RMSE comparable

[2]To reduce training sample bias, we use only 10% of the data and set a large prior hyperparameter (50).

to WLS across all factors. However, omitting the constraint leads to divergence from theory-implied values for traded factors, increasing bias and RMSE, underscoring the importance of the traded factor constraint. Nontraded factors are also affected; for instance, in the case of HKMcapital, BayesNC exhibits notably higher bias.

### 3.2 Simulation with Weak Factors

Weak factor problems are common in beta-pricing tests. Traditional two-pass methods produce invalid inferences when factors are weak. Our approach addresses this by excluding models with weak or irrelevant factors through marginal likelihood comparisons. Models with weak factors yield lower marginal likelihoods, despite risk premia appearing statistically significant. Such significance is misleading, as it does not improve pricing performance. Thus, weak factors are unlikely components of the correctly specified model, and their estimated risk premia are spurious.

We demonstrate in Section 3.1 that our estimate matches or surpasses WLS under the correct model specification. However, what if the factor is weak, particularly for nontraded factors? We conducted an additional simulation to test the model under a weak factor, where the beta is significantly smaller than the calibrated value. We still use the calibrated parameter set $[\boldsymbol{\mu}, \boldsymbol{\Sigma}_u, \boldsymbol{B}, \boldsymbol{\Sigma}_e]$ as described in Section 3.1 and each time simulate 7 factors that correspond to FF6 plus one nontraded factor (each column in Table 2). The sample period $T$ aligns with the empirical data, which equals 468 months. We derive returns from the 7-factor model and vary the beta scale using a parameter $w$.

$$\boldsymbol{r}_t = w\boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^T\boldsymbol{f}_t^T + w\boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t, \quad \boldsymbol{e}_t \sim \mathcal{N}(0, \boldsymbol{\Sigma}_e)$$

We multiply the $w$ on the parameter of nontraded factors' calibrated beta to measure weakness in the corresponding simulation run. A smaller $w$ means weaker, with a much smaller beta. We also consider different lambdas to check how the weak factor and strength of risk premia jointly affect the ML.

Table 2 reports results in two panels: Panel A sets the nontraded factor's risk premium to zero, while Panel B sets it to 0.01. Within each panel, three levels of factor

Table 2: **Simulation Evidence: Evaluating Weak Factors**

This table reports the log $BF$ (Bayes Factor), which denotes the log marginal likelihood difference between the traded-factor-only model and the model augmented with one nontraded factor. The baseline model incorporates six traded factors. Each column augments the baseline with a distinct nontraded factor, while each panel corresponds to varying levels of risk premia, parameterized by weakness ($w$). Lower $w$ indicates a smaller beta. Parameters are calibrated using true test assets and factors, with results averaged over 100 simulations.

| | log $BF$ | log $BF$ | log $BF$ | log $BF$ | log $BF$ | log $BF$ | log $BF$ | log $BF$ |
|---|---|---|---|---|---|---|---|---|
| | IndProd | LIQ | LTY | M2 | PCEDG | PCEND | HKMcapital | PEAR |
| | | | Panel A: $\lambda$ = 0 | | | | | |
| $w$=10 | 10966 | 8787 | 10493 | 8052 | 5594 | 12329 | 14315 | 5045 |
| $w$=1 | 103 | 34 | 80 | 20 | -37 | 146 | 182 | -49 |
| $w$=0.1 | -121 | -121 | -122 | -123 | -125 | -122 | -118 | -123 |
| | | | Panel B: $\lambda$ = 0.01 | | | | | |
| $w$=10 | 16763 | 8874 | 10654 | 21735 | 6055 | 17452 | 14005 | 5236 |
| $w$=1 | 311 | 36 | 86 | 555 | -33 | 344 | 180 | -43 |
| $w$=0.1 | -121 | -123 | -123 | -120 | -121 | -120 | -120 | -123 |

weakness are considered. A weakness level of 1 corresponds to the original calibrated beta, while 0.1 reduces beta to 1% of its original value, making it ten times smaller. We calibrate and evaluate nontraded factors individually from empirical data, each represented by a column. The log Bayes Factor (log $BF$) measures the difference in log ML between the FF6 model augmented with the nontraded factor and the baseline FF6 model. A positive log $BF$ indicates that including the nontraded factor improves pricing by increasing ML relative to the FF6 model.

Table 2 presents consistent findings: when the nontraded factor is sufficiently strong ($w \geq 1$), the model incorporating it (except for PCEDG and PEAR, whose calibrated $\boldsymbol{B}$ is weak) achieves a higher ML than the FF6-only model. This shows that a factor with zero risk premia can still enhance model fit if it has large betas. While it does not aid in explaining the cross section, it contributes to explaining the time series. However, as the factor weakens ($w < 1$), its ML falls below that of the FF6 model, despite the nontraded factor being part of the true data-generating process, even with a nonzero risk premium. This suggests that a factor with nonzero risk premia may not enhance ML if it is weak. The contribution of such a factor is offset by the penalty for including a weak factor.

Table 3: **Simulation Evidence: Evaluating Constant-Loading Factors**

This table reports the marginal likelihood of different models. The baseline model incorporates six traded factors. Different panels indicate different levels of risk premium $\lambda$. Different columns indicated different model specifications with Constant-Loading (CL) factors. Parameters are calibrated using true test assets and factors, with results averaged over 100 simulations.

| $\text{CLF}_1 + \text{CLF}_2$ | No CLF | $\text{CLF}_1$ | $\text{CLF}_2$ |
|---|---|---|---|
| Panel A: $\lambda = 0$ | | | |
| 120570 | 118844 | 118859 | **120646** |
| Panel B: $\lambda = 0.01$ | | | |
| 120561 | 118696 | 118824 | **120636** |

Another type of weak factor, referred to as a Constant-Loading (CL) factor, arises when $\boldsymbol{B}$ lacks sufficient cross-sectional variation. While a single CL factor is allowed (with the zero-beta rate given), issues may occur when there are at least two such factors. We show in Table 3 that we can effectively detect the case of more than one CL factor, and choose the model with only one CL factor (whose $\boldsymbol{B}$ is large enough). We simulate 8 factors that correspond to FF6 plus IndProd and LIQ, similarly to Section 3.1, but we name the two nontraded factors CLF since we will set the loadings of these two nontraded factors to be constant. The return is generated from

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^{T}\boldsymbol{f}_t^{T} + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t, \quad \boldsymbol{e}_t \sim \mathcal{N}(0, \boldsymbol{\Sigma}_e)$$

, where $\boldsymbol{f}_t^{NT} = [\text{CLF}_{1t}', \text{CLF}_{2t}']'$. $\text{CLF}_1$ and $\text{CLF}_2$ have the same loading $\boldsymbol{B}_{\text{CLF}_1} = \boldsymbol{B}_{\text{CLF}_2} = 0.1$ for all test assets. In that case, the true $\boldsymbol{B}$ matrix has reduced rank. Table 3 reports the ML of different models; all have six benchmark traded factors as in the above weak factor settings but differ in nontraded factors. No CLF stands for six traded factors only. Each panel corresponds to a distinct risk premium $\lambda$ level. No matter whether the $\lambda$ is zero or large (0.01), the model with two CLFs underperforms the model with only $\text{CLF}_2$, achieving a lower ML, indicating that scenarios with more than one CL factor are detectable and will be dominated by the model with only one CL factor. Since our model allows for one CL factor, the model with only $\text{CLF}_1$ and the model with only $\text{CLF}_2$ outperform the model with no CL factors.

### 3.3 Simulation for Model Comparison

Having shown that our approach effectively excludes weak factors by comparing models with and without them, we now focus on identifying the optimal model from a set of candidate factors. In our framework, varying factor sets produce distinct models, leading to different risk premia estimates for the same factor. Including redundant models is uninformative and may lead to errors under potential misspecification. We simulate and evaluate models within the specified factor set to address this.

We simulate factors and returns as in Section 3.1, but with more factors and a single combination generating returns. The factor set includes MktRf, SMB, HML, RMW, CMA, UMD, and several nontraded factors such as liquidity (LIQ), presidential economic approval rating (PEAR), nondurable consumption expenditure (PCEND), and intermediary capital ratio (HKMcapital). The true return-generating model comprises MktRf, SMB, HML, and HKMcapital, yielding $(2^6 - 1)(2^4) = 1008$ model specifications. Factors excluded from the true model are weak or irrelevant, implying $\boldsymbol{B} = \mathbf{0}$. The sample period $T$ matches the empirical data. We compute ML for all models, OLS and GLS CSR$^2$, and HJD.

We simulate 100 times, reporting the average log marginal likelihood, OLS GLS CSR$^2$, model rank by ML, and HJD across 100 runs. Additionally, we present the ML and average rank without the traded factor constraint (NC). Table 4 lists the top 10 models under the constraint, ranked by average ML. Only ML with the constraint correctly identifies the true model as top-ranked, with all other models' ML significantly lower. While CSR$^2$ rises with additional factors, the improvement is modest, rendering it unreliable for model selection. Consistent with CSR$^2$, adding factors reduces HJD. An average rank of 1 demonstrates that ML robustly identifies the true model among nearly 1,000 candidates when strong factors are present across 100 simulations. Performance deteriorates sharply without the traded factor constraint, with the true model only achieving an average rank of 9. In this case, marginal likelihood instead selects FF3 as the top model, underscoring the critical role of the traded factor constraint.

It is noteworthy that not all traded factors must be incorporated into the model,

Table 4: **Simulation Evidence: Model Comparison under Marginal Likelihood**

This table summarizes simulation results for 1008 model specifications with 10 factors: 6 traded and 4 nontraded. The true DGP is a four-factor model (MktRf, SMB, HML, HKMcapital), with parameters calibrated to the true test assets and factors. Reported metrics include log marginal likelihood, cross-sectional $R^2$ (%), average model rank by marginal likelihood, and Hansen-Jagannathan distance (HJD %). NC denotes the Bayesian model without traded factor constraints. Simulations are repeated 100 times.

| Top 10 marginal likelihood models | logML | logMLNC | rank | rankNC | $CSR^2$ | HJD |
|---|---|---|---|---|---|---|
| MktRf+SMB+HML+HKMcapital | **55306.7** | 55451.7 | **1.0** | 9.3 | 62.4 | 0.0169 |
| MktRf+SMB+HML+CMA+HKMcapital | 55245.7 | 55512.0 | 4.5 | 4.6 | 63.2 | 0.0166 |
| MktRf+SMB+HML+RMW+HKMcapital | 55245.7 | 55521.5 | 4.3 | 3.0 | 63.1 | 0.0166 |
| MktRf+SMB+HML+UMD+HKMcapital | 55245.4 | 55519.4 | 4.5 | 3.4 | 63.2 | 0.0166 |
| MktRf+SMB+HML+PEAR+HKMcapital | 55244.6 | 55497.1 | 4.8 | 6.6 | 63.3 | 0.0166 |
| MktRf+SMB+HML+PCEND+HKMcapital | 55244.1 | 55502.2 | 4.8 | 6.1 | 63.2 | 0.0167 |
| MktRf+SMB+HML+Liq+HKMcapital | 55244.1 | 55519.7 | 4.8 | 3.3 | 63.4 | 0.0165 |
| MktRf+SMB+HML | 55216.3 | 55678.9 | 8.8 | 1.0 | 47.9 | 0.0201 |
| MktRf+SMB+HML+CMA+UMD+HKMcapital | 55184.3 | 55413.6 | 16.0 | 18.8 | 64.0 | 0.0163 |
| MktRf+SMB+HML+RMW+CMA+HKMcapital | 55184.2 | 55420.5 | 16.0 | 15.1 | 63.8 | 0.0163 |

as evidenced by simulation results. We generate six traded factors with risk premia aligned to their respective factor means. Factors irrelevant to pricing are omitted, underscoring a key insight: the method effectively filters out weak nontraded and weak traded factors. Within this framework, three traded factors exhibit zero betas and are naturally classified as weak, which the top-performing model excludes.

Table IA.4 in the Internet Appendix presents a robustness check on the return covariance matrix. Specifically, we relax the diagonal assumption in the true data-generating process (DGP) by using the full covariance matrix to simulate returns, while maintaining the diagonal assumption in estimation. The results show that this misspecification does not affect model comparison: the true model is consistently identified as the top-ranked model in the simulations, as reported in Table IA.4.

## 4 Empirical Performance

This section summarizes our main findings. Section 4.1 introduces the data. Section 4.2 examines each nontraded factor. Section 4.3 presents model comparisons across all factor combinations. Section 4.4 provides an out-of-sample evaluation for the factor model implied investment performance.

### 4.1 Data

Our analysis considers a large cross section of test assets, 302 equity portfolios from Ken French's website. It consists of 25 portfolios sorted by size and book-to-market ratio, 17 industry portfolios, 25 portfolios sorted by operating profitability and investment, 25 portfolios sorted by size and variance, 35 portfolios sorted by size and net issuance, 25 portfolios sorted by size and accruals, 25 portfolios sorted by size and beta, 25 portfolios sorted by size and momentum, 25 portfolios sorted by size and operating profitability, 25 portfolios sorted by size and investment, 25 portfolios sorted by size and short-term reversal, and 25 portfolios sorted by size and long-term reversal. The sample ranges from January 1985 to December 2023.

We analyze 10 widely used traded factors: MktRf, SMB, HML, CMA, RMW, UMD, BAB, QMJ, IA, and ROE. The first six factors are sourced from Ken French's website, BAB and QMJ from AQR's website, and the final two from the global-q website. Additionally, we include 8 nontraded factors proposed in the literature: IndProd, LIQ, LTY, M2, PCEDG, PCEND, HKMcapital, and PEAR.[3]

Nontraded factors are typically represented by AR (autoregressive) model innovations rather than the raw factors (e.g., see Chen et al., 1986). For LIQ and HKMcapital, we directly retrieve data from the authors' websites. As these series are already processed as AR innovations in their respective studies, we use them without modification. For consumption growth factors like PCEDG and PCEND, we compute real per capita growth rates using population and CPI data from FRED, which is consistent with the literature (e.g., Yogo, 2006). For other level-based factors sourced from FRED or PEAR, we estimate AR(1) models to extract innovations as nontraded factors. Table IA.2 provides detailed definitions.

[3]Incorporating additional factors often renders exploring an extensive model space redundant. The initial set captures most traded factors, while many nontraded factors add limited explanatory power. For traded factors, as in Chib et al. (2024), one can exclude those priced by the eight-factor model and apply principal component analysis (PCA) to the residuals. Similarly, for nontraded factors, PCA can be applied to large sets (e.g., Giglio et al., 2025).

Table 5: **Evaluation of Single Nontraded Factors**

The table presents estimation results for individual nontraded factors, controlling for three benchmark traded factor models: CAPM, FF5, and All (FF6 + BAB + QMJ + IA + ROE). Panel A reports Bayesian estimates, while Panel B provides WLS results. The third row of each model in Panel A is log $BF$, representing the log marginal likelihood difference between the traded-factor-only model and the traded-factor-plus-one-nontraded-factor model. Mimicking portfolio estimates are shown in Panel C. 95% confidence (credible) intervals are in parentheses. The confidence interval for WLS is based on the Fama-Macbeth standard error.

| | IndProd | LIQ | LTY | M2 | PCEDG | PCEND | HKMcapital | PEAR |
|---|---|---|---|---|---|---|---|---|
| | | | | Panel A: Bayes | | | | |
| | -0.68 | 4.57 | 1.36 | -0.38 | 0.82 | -0.70 | 1.78 | -3.68 |
| CAPM | [-0.83,-0.53] | [3.56,5.56] | [0.74,1.96] | [-0.45,-0.32] | [0.50,1.13] | [-0.84,-0.56] | [1.13,2.36] | [-4.24,-3.09] |
| | -424 | -308 | 405 | -312 | 340 | -300 | 1291 | -22 |
| | 0.07 | -1.43 | 0.46 | 0.16 | -1.10 | -0.07 | 0.67 | 0.40 |
| FF5 | [-0.06,0.20] | [-2.31,-0.52] | [-0.43,1.34] | [ 0.09,0.24] | [-1.63,-0.55] | [-0.21,0.07] | [-0.03,1.35] | [-1.05,1.69] |
| | -332 | -251 | -351 | -427 | -122 | -184 | 1 | -239 |
| | -0.04 | -2.48 | 2.98 | 0.15 | 1.31 | -0.25 | 1.50 | 2.52 |
| All | [-0.17,0.09] | [-3.51,-1.45] | [ 2.15,3.75] | [ 0.08,0.22] | [ 0.45,1.99] | [-0.40,-0.11] | [ 0.76,2.24] | [ 1.55,3.40] |
| | -257 | -261 | -174 | -350 | -130 | -162 | -111 | -182 |
| | | | | Panel B: WLS | | | | |
| CAPM | -0.11 | 1.06 | -0.62 | -0.19 | -0.54 | -0.32 | 1.47 | -3.91 |
| | [-0.27,0.05] | [-1.00,3.12] | [-2.77,1.53] | [-0.38,0.00] | [-1.91,0.83] | [-0.60,-0.05] | [0.55,2.40] | [-6.06,-1.75] |
| FF5 | 0.04 | -0.81 | -0.54 | 0.07 | -0.61 | -0.06 | 0.30 | -0.13 |
| | [-0.11,0.20] | [-1.91,0.29] | [-1.78,0.71] | [-0.03,0.18] | [-1.37,0.15] | [-0.24,0.11] | [-0.57,1.18] | [-1.17,0.91] |
| All | -0.03 | -0.83 | 0.51 | 0.08 | -0.15 | -0.13 | 0.93 | 0.53 |
| | [-0.17,0.12] | [-1.83,0.18] | [-0.56,1.59] | [ 0.00,0.17] | [-0.84,0.55] | [-0.29,0.03] | [ 0.09,1.77] | [-0.31,1.37] |
| | | | | Panel C: Mimicking | | | | |
| Mimicking | -0.20 | 0.02 | -0.46 | 0.01 | 0.10 | -0.07 | 1.42 | 0.47 |
| | [-0.27,-0.12] | [-0.46,0.49] | [-0.97,0.06] | [-0.03,0.04] | [-0.13,0.32] | [-0.15,0.02] | [0.83,2.02] | [0.16,0.79] |

## 4.2 Testing Individual Nontraded Factors

Before evaluating all the factors of interest as a full set, we first assess the weakness of each nontraded factor while controlling for different sets of traded factors. Specifically, we analyze three benchmark traded factor models: CAPM, FF5, and All traded factors (All). Panel A of table 5 presents the estimated risk premia, 95% credible interval, and log $BF$ for each nontraded factor. For instance, the first row provides the estimated risk premium, while the second row shows the 95 % credible interval. The third row of "log $BF$" indicates the difference in logML between two models with and without that nontraded factor. For example, for IndProd under MktRf control, log $BF$ less than zero means that the model includes both MktRf and IndProd under-performs the model with only MktRf, while the full set of factors includes both MktRf and IndProd. Panel B gives the estimation of WLS, and Panel C is the mimicking portfolio estimate with a 95% confidence interval in parentheses.

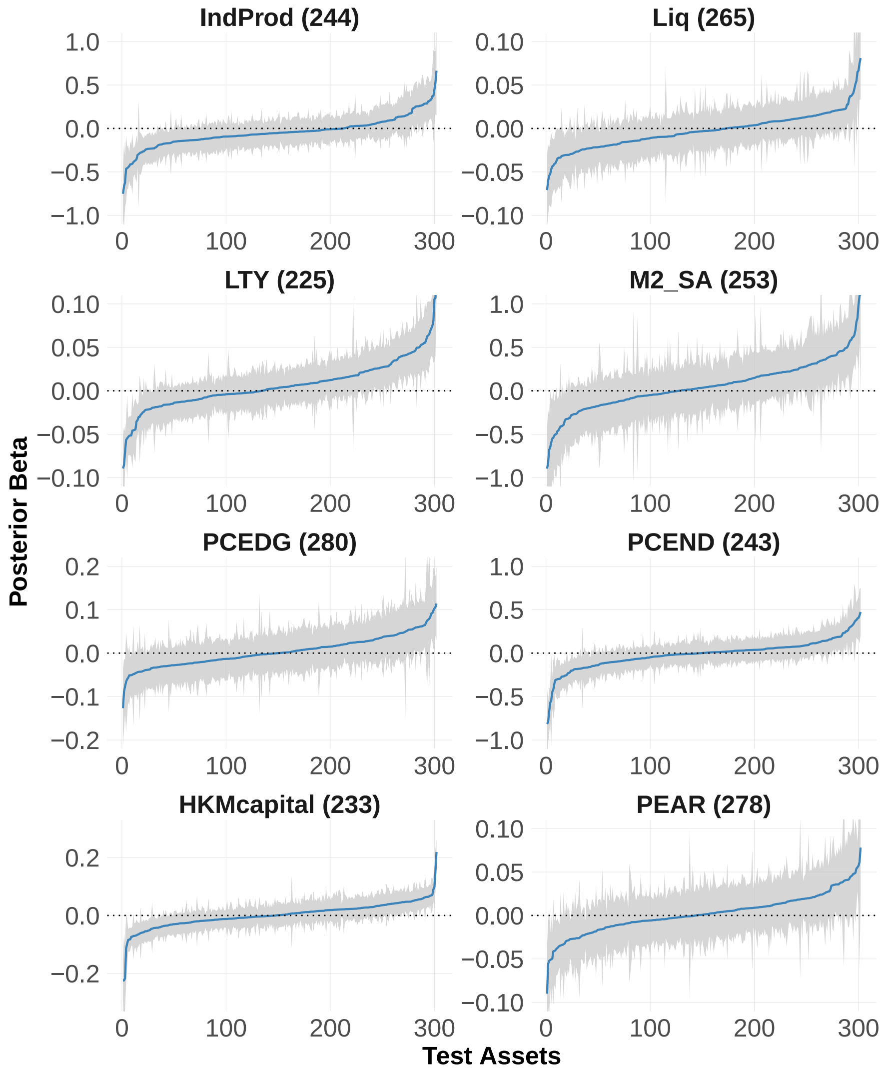


Figure 1: Posterior Cross-sectional Beta for Each Nontraded Factor
This figure shows the posterior beta for each nontraded factor in Table 5 (last row of Panel A) under All traded factor control along with its 95% credible interval, represented by the grey shaded area. Beta is cross-sectionally sorted in ascending order. The number of betas for each factor that is not significantly different from 0 is in parentheses.

IndProd, LIQ, M2, PCEND, and PEAR seem the least effective of all the nontraded factors, as they consistently fall short compared to the pure traded factor models across all evaluation benchmarks. LTY and PCEDG marginally enhance the CAPM, but the contribution diminishes in the remaining benchmark cases. Out of 8 nontraded factors, HKMcapital seems to be the most prominent. Its risk premium estimate is relatively significant and aligns closely with the WLS estimate and the mimicking portfolio. It is the only nontraded factor that improves the FF5 benchmark; however, it still lacks relevance compared to all traded factors. This suggests that, while it performs better than other weaker nontraded factors, it may still not be considered conclusively useful. Most of these factors show significant risk premia estimates, but these results are likely spurious due to their inherent weaknesses.

To highlight factor weaknesses, Figure 1 shows the cross-sectional posterior $\boldsymbol{B}^{NT}$ (last row of Panel A, Table 5) with 95% credible intervals. The number of $\boldsymbol{B}^{NT}$ estimates not significantly different from zero is in parentheses. Among 302 test assets, over 200 $\boldsymbol{B}^{NT}$ are insignificant for all eight nontraded factors, ranging from 225 for LTY to 280 for PCEDG. Although some factors, like M2, show cross-sectional variation in $\boldsymbol{B}^{NT}$, their credible intervals remain wide. These results confirm the spurious nature of risk premia estimates in Table 5. With such weak betas, identifying risk premia is difficult. The next section offers a comprehensive comparison of these factors.

Table 6: **Top 5 Models Ranked by Marginal Likelihood**

This table reports the top 5 models from all possible combinations of 18 factors, out of all possible combinations of 18 factors. The log marginal likelihood, Cross-Sectional $R^2$ (%), Hansen-Jagannathan Distance (HJD), and model probability are presented.

| Rank | Top 5 Model | logML | CSR$^2$ | HJD | Prob |
|---|---|---|---|---|---|
| 1 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ | **322024** | 49.03 | 0.0071 | **1** |
| 2 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + ROE | 322008 | 49.68 | 0.0071 | 0 |
| 3 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + HKMcapital | 321905 | 49.10 | 0.0071 | 0 |
| 4 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + ROE + HKMcapital | 321895 | 49.74 | 0.0071 | 0 |
| 5 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + PCEDG | 321891 | 49.10 | 0.0071 | 0 |

### 4.3 Joint Model Comparison

The previous section shows that most nontraded factors barely contribute. We now compare all possible model combinations, considering 10 traded and 8 nontraded factors. CAPM serves as the baseline, with the market factor always included, as excluding MktRf is hard to justify given a broad cross section of test assets (though this constraint can be relaxed). This yields $2^{17}$ models for evaluation.

Table 6 reports the top 5 marginal likelihood models, the CSR$^2$ and HJD.[4] Unsurprisingly, the top model includes only traded factors, which aligns with the findings from the previous section. However, not all traded factors are selected—for instance, IA and ROE are excluded. Imposing the traded factor constraint does not guarantee their selection. The CSR$^2$ is 49.03%, which is not the highest, while the HJD is 0.0071, almost the same among the top 5 models. Adding more factors will increase CSR$^2$, although the increase is typically only about 0.5%. Tables IA.6 and IA.7 in the appendix provide robustness checks of different hyperparameters in the training sample, and the top model remains the same.

To see how the resulting model is excluded by ML, we take the rank 2 model as an example since it only adds one factor, ROE, into the rank 1 model. Figure 2 plots the cross-sectional posterior $\boldsymbol{B}$ and the 95% credible intervals for each factor in the rank 2 model, with the number of betas that is not significantly different from zero in parentheses. Figure IA.2 in the Internet Appendix gives the plot for the rank 1 model. The market factor is exceptionally strong, with all $\boldsymbol{B}$ significantly different from zero. The remaining 7 factors in the rank 1 model are weaker but at least have about half of $\boldsymbol{B}$ different from zero out of 302 test assets. The ROE, which is excluded from the rank 1 model, has a significantly larger proportion of $\boldsymbol{B}$ values—over two-thirds—that are not significantly different from zero. This results in overparameterization, leading to a penalty in the ML, thus being excluded from the rank 1 model.

Panel A of Table 7 shows the estimation of risk premia in the rank 1 models. The rows provide the posterior mean, standard deviation, and credible intervals for the Bayesian approach, along with the mean, standard error, and confidence intervals

[4] We compute the model probability in the final column. We assign equal prior probabilities without a reason to favor one model ex-ante. The posterior probability of model $\mathcal{M}$ is $P(\mathcal{M} \mid Data) = \frac{\mathrm{ML}(Data|\mathcal{M})P(\mathcal{M})}{\sum_{l=1}^{2^{17}} \mathrm{ML}(Data|\mathcal{M}_l)P(\mathcal{M}_l)}$. We apply model averaging as in Avramov et al. (2023). Given the top model's substantially higher ML, it effectively dominates with a probability of 1 (rounded to six decimal places). Thus, model averaging reduces to selecting the top model.

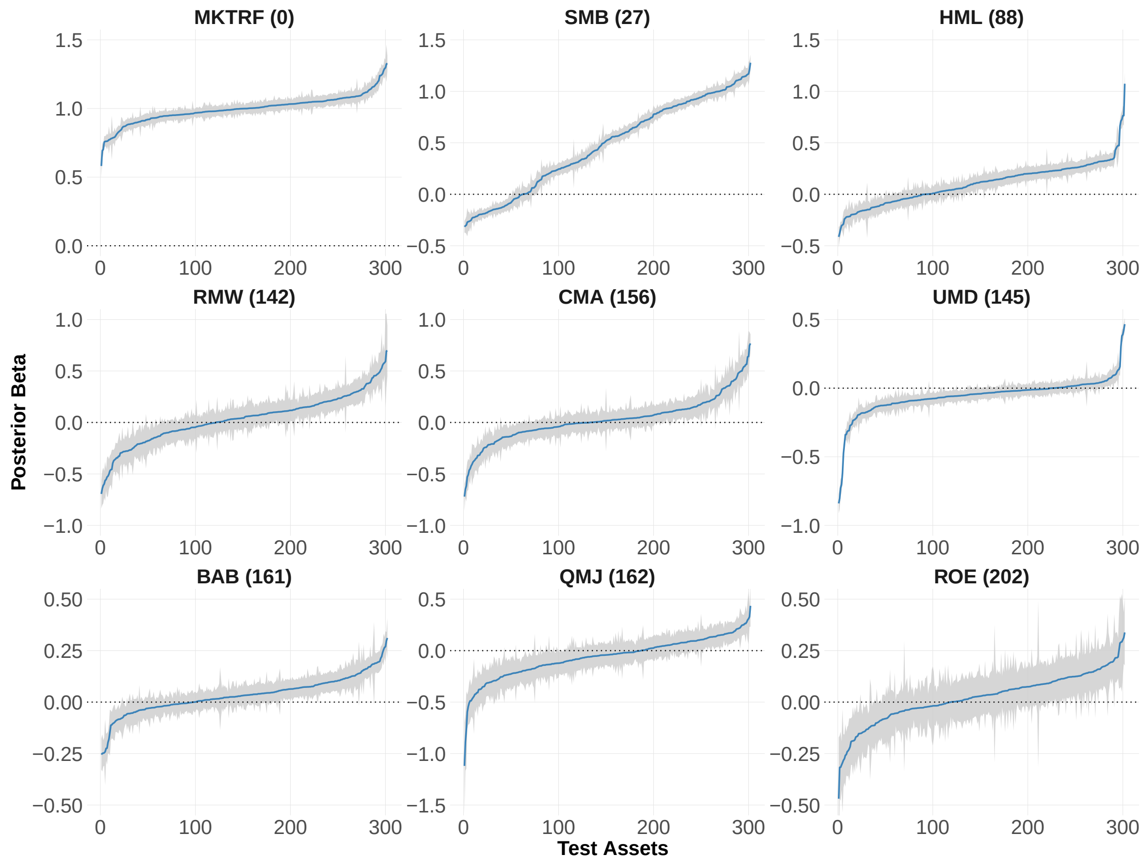


Figure 2: Posterior Cross-sectional Beta in the Rank 2 Model
This figure shows the posterior beta for each factor in the rank 2 model in Table 6 along with its 95% credible interval, represented by the grey shaded area. Beta is cross-sectionally sorted in ascending order. The number of betas for each factor that is not significantly different from 0 is in parentheses.

for WLS and mimicking portfolio estimates. Since there are no nontraded factors, the risk premia estimates for the eight traded factors are closely aligned with their means. Five factors are significant in Bayesian estimation, while four are significant under WLS and mimicking portfolio estimation. Although not all selected factors are significant, as discussed earlier, significance is not guaranteed. However, all selected factors contribute to explaining the time-series variation, with sufficient loadings, as shown in Figure IA.2 in the Internet Appendix. Panel B presents the risk price ($\gamma$) estimates in the rank 1 model. MktRf, CMA, BAB, and QMJ not only exhibit significant risk premia but also demonstrate significant SDF loadings.

Figure 3 presents factor selection frequencies across the top 10, 20, and 30 models, emphasizing their relative significance. Traded factors like MktRf, SMB, and HML

Table 7: **Risk Premia and Risk Price Estimation in Rank 1 Model**

Panel A of this table presents the risk premia estimates for the top-performing model (see table 6) among all combinations of 18 factors. Estimates are reported using Bayesian, WLS, and mimicking portfolio approaches. For Bayesian estimation, we provide the posterior mean, standard deviation, and the 95% credible interval (lower and upper bounds). For WLS and mimicking portfolios, we report the mean, Fama-Macbeth standard error, and 95% confidence interval. Panel B presents the Bayesian estimation of risk price. All values are expressed in percentage terms.

| | MktRf | SMB | HML | RMW | CMA | UMD | BAB | QMJ |
|---|---|---|---|---|---|---|---|---|
| | | | | Panel A: Risk Premia | | | | |
| | | | | Bayes | | | | |
| postmean | 0.74 | 0.10 | 0.15 | 0.40 | 0.21 | 0.41 | 0.79 | 0.45 |
| postsd | 0.22 | 0.14 | 0.14 | 0.11 | 0.10 | 0.21 | 0.17 | 0.12 |
| lower | 0.31 | -0.17 | -0.13 | 0.19 | 0.02 | -0.03 | 0.45 | 0.21 |
| upper | 1.16 | 0.36 | 0.42 | 0.61 | 0.40 | 0.83 | 1.11 | 0.67 |
| | | | | WLS | | | | |
| mean | 0.77 | 0.10 | 0.14 | 0.31 | 0.15 | 0.48 | 0.45 | 0.24 |
| std.err | 0.21 | 0.14 | 0.15 | 0.13 | 0.11 | 0.21 | 0.22 | 0.14 |
| lower | 0.36 | -0.18 | -0.16 | 0.06 | -0.06 | 0.06 | 0.01 | -0.03 |
| upper | 1.18 | 0.37 | 0.44 | 0.55 | 0.36 | 0.90 | 0.89 | 0.51 |
| | | | | Mimicking | | | | |
| mean | 0.80 | 0.09 | 0.10 | 0.42 | 0.22 | 0.46 | 0.20 | 0.28 |
| std.err | 0.21 | 0.14 | 0.15 | 0.11 | 0.10 | 0.21 | 0.16 | 0.12 |
| lower | 0.39 | -0.17 | -0.19 | 0.19 | 0.03 | 0.05 | -0.12 | 0.06 |
| upper | 1.21 | 0.36 | 0.38 | 0.64 | 0.42 | 0.87 | 0.52 | 0.51 |
| | | | | Panel B: Bayes Risk Price | | | | |
| postmean | 10.20 | 4.56 | -2.36 | -4.42 | 9.10 | -0.82 | 5.47 | 19.88 |
| postsd | 1.54 | 1.99 | 2.44 | 3.70 | 3.56 | 1.40 | 1.72 | 3.97 |
| lower | 7.35 | 0.59 | -7.21 | -11.83 | 2.26 | -3.55 | 2.18 | 12.11 |
| upper | 13.33 | 8.46 | 2.18 | 2.88 | 16.38 | 1.79 | 8.88 | 27.68 |

consistently dominate, appearing in nearly all top 30 models, underscoring their robust explanatory power. In contrast, ROE and IA exhibit low selection rates, with IA included even less often than some nontraded factors. Among nontraded factors, HKMcapital achieves the highest inclusion rate, slightly outperforming others. Conversely, LIQ and M2 show minimal inclusion, indicating limited relevance. These results highlight the stronger explanatory power of traded factors relative to nontraded factors within the beta-pricing framework.

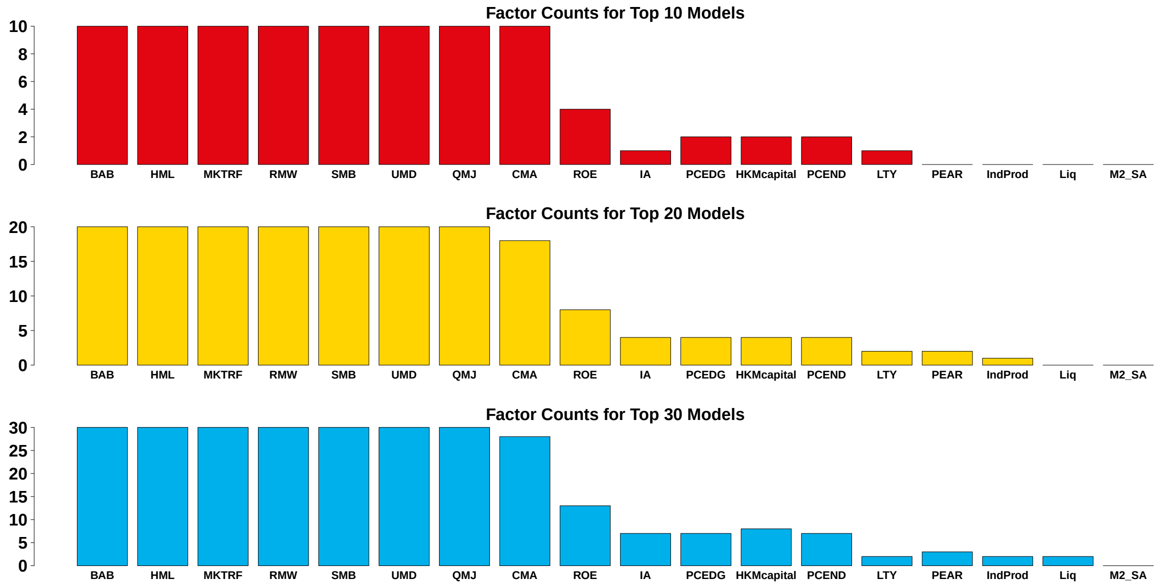


Figure 3: Factor Selection Count

This figure presents a bar plot illustrating the frequency of factor selection among the top 10, 20, and 30 models derived from all possible combinations of 18 factors.

## 4.4 Factor Model Implied Investment Performance

By selecting the optimal model from a set of candidate factor specifications, we evaluate its role in constructing the tangency portfolio for a comprehensive set of test assets. A distinctive strength of our framework is its ability to estimate the model-implied expected returns and the covariance matrix within the context of the chosen optimal asset pricing factor model. Our approach leverages Bayesian model estimation, seamlessly incorporating parameter and model uncertainty into estimating model-implied expected returns and the covariance matrix. These elements are pivotal for robust portfolio construction (e.g., Feng and He, 2022).

We split the sample period into an in-sample (INS) period (first half) and an out-of-sample (OOS) period (remaining months). For each model estimation, including top models, CAPM, FF3, FF5, and all-factor models, the expected return implied by the selected model is computed as $\boldsymbol{\mu}_{tan} = \boldsymbol{B}[\boldsymbol{\mu}^T, \boldsymbol{\lambda}^{NT}]$ (or $\boldsymbol{\mu}_{tan} = \boldsymbol{B}\boldsymbol{\lambda}$ for WLS). The inverse covariance matrix is given by:

$$\Sigma_{tan}^{-1} = \Sigma_e^{-1} - \Sigma_e^{-1}\boldsymbol{B}(\Sigma_u^{-1} + \boldsymbol{B}'\Sigma_e^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}'\Sigma_e^{-1},$$

Table 8: **Out-of-Sample Model Performance**

Panel A reports annualized Sharpe Ratios (SR) and maximum drawdown (MDD) for tangency portfolios derived from seven factor models, using out-of-sample (OOS) observations. Portfolio weights are estimated via Bayes and WLS methods. Weights are computed once using in-sample (INS) data (first half) and applied to OOS returns. Test assets include all test portfolios. Panel B summarizes results for equal-weighted and mean-variance efficient portfolios.

| Panel A : Model-implied | | | | |
|---|---|---|---|---|
| | Bayes SR | WLS SR | Bayes MDD | WLS MDD |
| Rank1 | 0.983 | 0.600 | 0.231 | 0.428 |
| Rank2 | 0.975 | 0.613 | 0.246 | 0.407 |
| Rank3 | 0.878 | 0.588 | 0.307 | 0.445 |
| CAPM | 0.568 | 0.568 | 0.482 | 0.482 |
| FF3 | 0.650 | 0.559 | 0.428 | 0.437 |
| FF5 | 0.883 | 0.796 | 0.256 | 0.246 |
| ALL | 0.749 | 0.813 | 0.560 | 0.384 |
| Panel B: EW and MVE | | | | |
| EW SR | MVE SR | | EW MDD | MVE MDD |
| 0.526 | 0.566 | | 0.491 | 0.472 |

where $\Sigma_e^{-1}$ is diagonal. This approach leverages the factor covariance structure to avoid inverting high-dimensional matrices. The tangency portfolio weights are derived as $\frac{\Sigma_{tan}^{-1}\boldsymbol{\mu}_{tan}}{\mathbf{1}^T\Sigma_{tan}^{-1}\boldsymbol{\mu}_{tan}}$. Model estimation is conducted on INS data, while OOS data is used to construct portfolios based on these weights. Performance is assessed using OOS tangency portfolio Sharpe ratios.

Table 8 presents model performance based on Sharpe Ratios of tangency portfolios derived from test assets for both Bayes and WLS frameworks. The rank 1, rank 2, and rank 3 models are detailed in Table 6. Panel A summarizes results for various factor models used to estimate portfolio weights, which are then applied to construct tangency portfolios. The weights are estimated from the initial half of the data and applied to the rest of the OOS return. Panel B reports the OOS results for equal-weighted (EW) and mean-variance efficient (MVE) portfolios. For MVE, we employ the diagonal covariance matrix of in-sample returns to ensure invertibility.

Considering the Bayesian results, the top model demonstrates superior tangency portfolio weights, achieving annualized Sharpe Ratios of 0.983. It substantially outperforms standard CAPM, FF3, and FF5 models. Notably, incorporating additional factors, as indicated in the "ALL" row, does not improve performance. Moreover, it significantly surpasses equal-weighted and mean-variance efficient portfolios, which yield OOS annualized Sharpe Ratios of 0.526 and 0.566, respectively. In contrast, WLS results favor all factors, yet their Sharpe Ratios remain below those of the Bayesian top model. These findings highlight the robustness of our Bayesian framework for model comparison and estimation.

Assessing the downside risk of the tangency portfolio is essential. Table 8 also reports the maximum drawdown (MDD) for the constant-weight tangency portfolios. MDD over any overlapping one-year period is defined as:

$$MDD = \max_{1 \leq t_1 \leq t_2 \leq T} \left(Y_{t_1} - Y_{t_2}\right), \text{ s.t. } |t_2 - t_1| \leq 12,$$

where $Y_{t_1}$ and $Y_{t_2}$ represent cumulative log returns from month $0$ to $t_1$ and $t_2$, respectively, with a maximum span of 12 months. The results show that the top-performing model under Bayesian estimation attains the lowest MDD, consistently outperforming the WLS, EW, and MVE benchmarks while reducing downside risk. Results for the FF25 and 17 Industry portfolios appear in the Internet Appendix in Tables IA.8 and IA.9. The rank-2 model performs best for FF25, whereas the all-factor model leads for IND17. However, the rank-1 model still remains the second-best out-of-sample performer for both subsets. Overall, the out-of-sample results confirm that models selected via our Bayesian ML framework deliver superior economic explanatory power for test asset returns, as reflected in improved model-implied mean and covariance estimates for tangency portfolio construction.

# 5 Revisiting CSR$^2$ Measure and Consumption-based Models

## 5.1 Revisiting CSR$^2$ Measure in Kan, Robotti, and Shanken (2013)

Kan, Robotti, and Shanken (2013) assess asset pricing performance using CSR$^2$ and derive its asymptotic distribution. They find that many large CSR$^2$ differences lack statistical significance. We reexamine their findings through our framework. There are eight models[5]: CAPM, Conditional CAPM (C-LAB) of Jagannathan and Wang (1996), FF3, ICAPM of Petkova (2006), unconditional consumption model (CCAPM), CC-CAY of Lettau and Ludvigson (2001), U-CCAPM of Parker and Julliard (2005), and D-CCAPM of Yogo (2006). There are 14 factors: two models with traded factors, three with nontraded factors, and three combining both.

We employ the dataset from the original study, covering February 1959 to July 2007, as described in Appendix IA.I.1. Table 9 reports estimation results, logMLs, and CSR$^2$ for these 8 models.[6] When the zero-beta rate is fixed at the risk-free rate, we analyze excess returns following Kan et al. (2013). Consistent with their approach, the reported CSR$^2$ is the uncentered measure under the specified zero-beta rate.[7] The OLS and GLS estimates, along with CSR$^2$, exactly align with their results.[8]

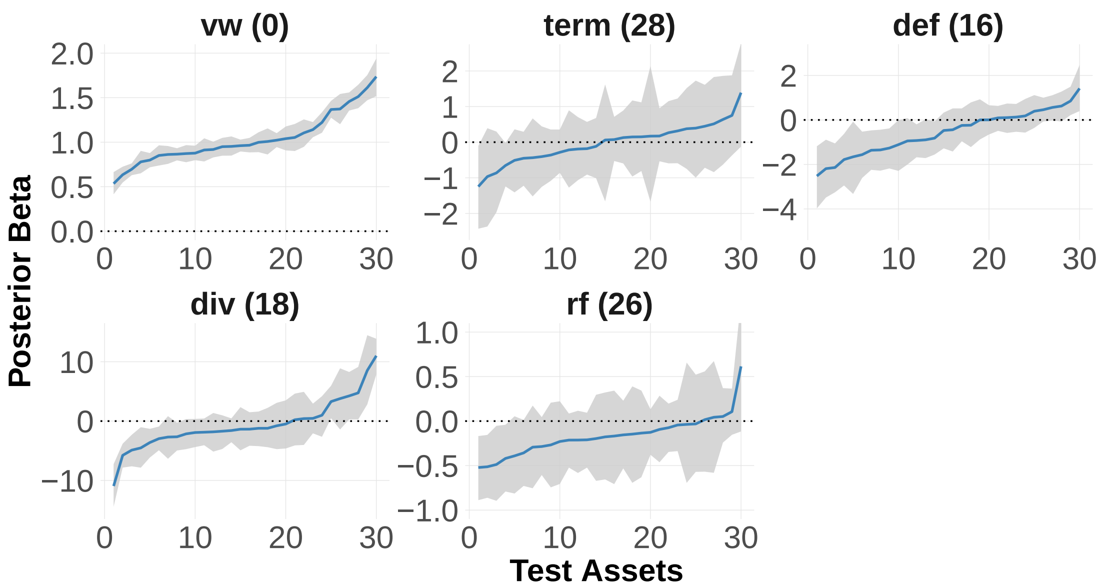


Figure 4: Posterior Cross-sectional Beta for ICAPM in Kan et al. (2013)

This figure shows the posterior beta for ICAPM in Table 9 along with its 95% credible interval, represented by the grey shaded area. Beta is cross-sectionally sorted in ascending order. The number of betas for each factor that is not significantly different from 0 is in parentheses.

[5]Each model is characterized by implied cross-sectional restrictions and is reformulated using the economically constrained time-series framework detailed in the Internet Appendix IA.I.1.

[6]Since there are many nontraded factors in their study, the training sample prior may not be stable because of the covariance structure. We use a 50 hyperparameter to make the prior more diffuse.

[7]Using uncentered CSR$^2$ does not alter relative rankings but ensures positivity.

[8]GLS CSR$^2$ employs a weighting matrix based on the inverse covariance matrix of returns, while GLS CSR$^2_{\text{res}}$ uses the inverse residual covariance matrix.

Table 9: **Comparison of 8 Models in Kan, Robotti, and Shanken (2013)**

This table summarizes the eight models from Kan et al. (2013). The models are estimated using monthly returns of the 25 ME-B/M portfolios and five industry portfolios, covering February 1959 to July 2007. For Bayesian estimation, we present the posterior mean, standard deviation, and 95% credible interval bounds. For OLS and GLS, we report the mean and Fama-Macbeth standard error. All figures are in percentages. Each model's log ML and uncentered $CSR^2$ are reported. GLS $CSR^2$ employs the return covariance matrix as a weight (Kan et al., 2013), while $CSR^2_{res}$ uses the residual covariance matrix.

| | | Bayes | | | | OLS | | GLS | |
|---|---|---|---|---|---|---|---|---|---|
| | | postmean | postsd | lower | upper | mean | se | mean | se |
| Rank 1 : FF3 | *vw* | 0.41 | 0.19 | 0.02 | 0.80 | 0.50** | 0.18 | 0.51** | 0.18 |
| | *smb* | 0.17 | 0.15 | -0.12 | 0.44 | 0.16 | 0.13 | 0.23 | 0.13 |
| | *hml* | 0.44 | 0.13 | 0.19 | 0.69 | 0.39** | 0.12 | 0.41** | 0.12 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 64219 | | | | 95.78 | | 27.41 | 96.97 |
| Rank 2 : D_CCAPM | *vw* | 0.41 | 0.19 | 0.03 | 0.78 | 0.55** | 0.18 | 0.50** | 0.18 |
| | *cg* | 1.10 | 0.18 | 0.77 | 1.48 | 0.93** | 0.25 | 0.16 | 0.12 |
| | *cgdur* | 0.49 | 0.41 | -0.29 | 1.26 | 0.00 | 0.64 | 0.62 | 0.39 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 57095 | | | | 88.33 | | 8.32 | 90.63 |
| Rank 3 : CAPM | *vw* | 0.41 | 0.19 | 0.04 | 0.80 | 0.63** | 0.19 | 0.50** | 0.18 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 57082 | | | | 85.78 | | 5.82 | 90.36 |
| Rank 4 : C_LAB | *vw* | 0.41 | 0.19 | 0.03 | 0.79 | 0.57** | 0.18 | 0.51** | 0.18 |
| | *lab* | -0.15 | 0.17 | -0.47 | 0.22 | -0.20 | 0.13 | -0.12** | 0.06 |
| | *prem* | 0.37 | 0.08 | 0.20 | 0.54 | 0.40** | 0.13 | 0.02 | 0.07 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 57033 | | | | 89.27 | | 9.06 | 90.70 |
| Rank 5 : ICAPM | *vw* | 0.40 | 0.20 | 0.00 | 0.79 | 0.53** | 0.18 | 0.52** | 0.18 |
| | *term* | 0.22 | 0.06 | 0.10 | 0.35 | 0.31** | 0.08 | 0.24** | 0.05 |
| | def | -0.11 | 0.04 | -0.19 | -0.02 | -0.10 | 0.06 | -0.07 | 0.04 |
| | div | -0.05 | 0.01 | -0.07 | -0.02 | -0.06** | 0.01 | -0.04** | 0.01 |
| | *rf* | -0.49 | 0.13 | -0.75 | -0.24 | -0.59** | 0.14 | -0.42** | 0.10 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 56958 | | | | 97.24 | | 33.90 | 93.32 |
| Rank 6 : CCAPM | *cg* | 0.58 | 0.06 | 0.47 | 0.70 | 0.67** | 0.20 | 0.26** | 0.11 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 46983 | | | | 87.96 | | 4.42 | 4.83 |
| Rank 7 : U_CCAPM | *cg36* | 3.78 | 0.40 | 3.03 | 4.60 | 4.67** | 1.29 | 1.95** | 0.51 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 46877 | | | | 94.59 | | 11.01 | 11.79 |
| Rank 8 : CC_CAY | *cg* | 0.42 | 0.13 | 0.18 | 0.67 | 0.35** | 0.16 | 0.27** | 0.11 |
| | cay | 0.48 | 0.36 | -0.24 | 1.21 | 0.67 | 0.41 | 0.73** | 0.26 |
| | cgcay | 0.00 | 0.00 | 0.00 | 0.01 | 0.01** | 0.00 | 0.00 | 0.00 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 46864 | | | | 88.57 | | 10.53 | 11.26 |

Their top-performing model, ICAPM, has the highest OLS $CSR^2$ up to 97.24, although it's not significantly higher than FF3 (95.78). In fact, ICAPM's $CSR^2$ doesn't significantly dominate any other models according to Kan et al. (2013)'s test(with zero-beta rate given, their Table IA.III). The outperformance of $CSR^2$ can be attributed to its inclusion of the largest number of factors. Regarding marginal likelihood, FF3 demonstrates the best performance, significantly outperforming the others, while the ML for ICAPM is much lower, even lower than that of the single-factor CAPM. Figure 4 plots the posterior beta of the ICAPM with its posterior credible interval. Consistent with our main results, apart from the only traded factor, market *vw*, a large portion of the betas for the remaining four nontraded factors are not significantly different from zero. Although the risk premia estimations for *term*, *div*, and *rf* are significant, they are unreliable because these factors explain little time-series variation due to their limited exposure. Beyond the traded-factor-only model FF3, D-CCAPM, which considers both nondurable and durable consumption, achieves the second-best model. Section 5.2 also reviews multiple consumption measures in detail.

In addition to 8 pre-specified models, we consider all possible model combinations of those 14 factors, including combinations of traded and nontraded factors. Table 10 shows the top 5 models with the highest logML. The estimation of risk premia is conducted using Bayes, OLS, and GLS, with logML and $CSR^2$ reported. Since all models contain FF3 with additional nontraded factors, we only report risk premia estimates for nontraded factors in different models.

Consistent with our main empirical findings, the pure traded-factor FF3 model remains superior among all factor combinations in the marginal likelihood analysis of Kan et al. (2013). The second-best model incorporates a nontraded factor, *div* (market dividend yield), derived from the ICAPM. However, its log marginal likelihood is substantially lower than that of FF3, despite its significant risk premium under Bayes, OLS, and GLS estimations. Models including other nontraded factors rank below FF3. We therefore conclude that nontraded factors contribute minimal incremental statistical value beyond the traded-factor model.

Table 10: **Comparison of All Model Specification**

This table presents the top 5 models among all combinations of 14 factors analyzed in Kan et al. (2013). For Bayesian models, we report the posterior mean, standard deviation, and the 95% credible interval bounds. For OLS and GLS, we provide the mean and Fama-Macbeth standard error. All figures are in percentages. Each model includes FF3, so we report only the risk premia estimates for the nontraded factor. Additionally, we present the log ML and uncentered CSR$^2$. GLS CSR$^2$ uses the return covariance matrix as a weighting matrix (Kan et al., 2013), while CSR$^2_{\text{res}}$ employs the residual covariance matrix.

| | | Bayes | | | | OLS | | GLS | |
|---|---|---|---|---|---|---|---|---|---|
| | | postmean | postsd | lower | upper | mean | se | mean | se |
| Rank 1 | FF3 | logML | | | | CSR$^2$ | | CSR$^2$ | CSR$^2_{res}$ |
| | | 64219 | | | | 95.78 | | 27.41 | 96.97 |
| Rank 2 | FF3 + *div* | -0.03 | 0.01 | -0.04 | -0.01 | -0.03** | 0.01 | -0.02** | 0.01 |
| | | logML | | | | CSR$^2$ | | CSR$^2$ | CSR$^2_{res}$ |
| | | 64207 | | | | 96.09 | | 27.9 | 96.99 |
| Rank 3 | FF3 + *cg36* | 2.80 | 0.54 | 1.81 | 3.95 | 2.13** | 0.70 | 0.60 | 0.58 |
| | | logML | | | | CSR$^2$ | | CSR$^2$ | CSR$^2_{res}$ |
| | | 64201 | | | | 96.22 | | 27.92 | 96.99 |
| Rank 4 | FF3 + *lab* | -0.33 | 0.07 | -0.47 | -0.21 | -0.33** | 0.08 | -0.16** | 0.06 |
| | | logML | | | | CSR$^2$ | | CSR$^2$ | CSR$^2_{res}$ |
| | | 64188 | | | | 96.91 | | 32.67 | 97.19 |
| Rank 5 | FF3 + *div* | -0.02 | 0.01 | -0.04 | 0.00 | -0.03** | 0.01 | -0.02** | 0.01 |
| | + *cg36* | 2.81 | 0.57 | 1.77 | 4.00 | 2.37** | 0.72 | 0.66 | 0.59 |
| | | logML | | | | CSR$^2$ | | CSR$^2$ | CSR$^2_{res}$ |
| | | 64187 | | | | 96.65 | | 28.54 | 97.02 |

### 5.2 Revisiting Consumption-based Model

In Sections 4 and 5.1, we assess consumption-based factor models, which rely on NIPA-reported measures (National Income and Product Accounts). However, these measures show weak correlations with test asset returns and are considered weak factors. The literature explores alternative consumption proxies, with Kleibergen and Zhan (2020) introducing robust tests to estimate factor risk premia. To illustrate our Bayesian approach, we revisit their study to estimate and assess these weak consumption factors using marginal likelihood.

We analyze five consumption measures following Kleibergen and Zhan (2020): (i) NIPA's consumption expenditure on nondurable goods and services ("Reported"),

Table 11: **Estimation and Marginal Likelihood under MktRf Control**

This table summarizes five consumption models from Kleibergen and Zhan (2020). The test assets are 30 portfolios sorted by size, value, and investment, using annualized data from 1960 to 2014. For the Bayesian estimates, we report the posterior mean, standard deviation, and 95% credible interval. For OLS and GLS, we present the mean and Fama-Macbeth standard error. All figures are in percentages. Each model includes the log marginal likelihood and uncentered $CSR^2$. GLS $CSR^2$ uses the return covariance matrix as weights, while $CSR^2_{res}$ employs the residual covariance matrix.

| | | Bayes | | | | OLS | | GLS | |
|---|---|---|---|---|---|---|---|---|---|
| | | postmean | postsd | lower | upper | mean | se | mean | se |
| Rank 1 | MktRf | 5.00 | 2.66 | -0.12 | 10.11 | 5.96** | 2.40 | 4.71** | 2.29 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 1824 | | | | 91.94 | | 6.6 | 83.75 |
| Rank 2 | MktRf | 5.06 | 2.59 | -0.05 | 9.99 | 6.59** | 2.42 | 4.76** | 2.29 |
| | Unfiltered | 1.79 | 0.59 | 0.69 | 2.99 | 2.21** | 0.84 | 0.78 | 0.48 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 1809 | | | | 95.01 | | 9.45 | 84.31 |
| Rank 3 | MktRf | 5.08 | 2.58 | -0.04 | 9.98 | 7.03** | 2.46 | 4.74** | 2.29 |
| | Q4-Q4 | 1.19 | 0.32 | 0.60 | 1.88 | 1.87** | 0.56 | 0.31 | 0.26 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 1808 | | | | 96.9 | | 7.58 | 84.13 |
| Rank 4 | MktRf | 5.03 | 2.55 | -0.16 | 9.75 | 6.58** | 2.45 | 4.74** | 2.29 |
| | Garbage | 2.37 | 0.65 | 1.20 | 3.74 | 4.06** | 1.37 | 0.84 | 0.43 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 1796 | | | | 95.27 | | 8.25 | 84.09 |
| Rank 5 | MktRf | 5.07 | 2.58 | -0.05 | 9.94 | 7.89** | 2.56 | 4.91** | 2.30 |
| | P-J | 2.35 | 0.62 | 1.20 | 3.60 | 4.41** | 1.34 | 1.22** | 0.59 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 1783 | | | | 97.07 | | 11.84 | 85.9 |
| Rank 6 | MktRf | 5.08 | 2.58 | -0.03 | 10.04 | 7.00** | 2.44 | 4.69** | 2.29 |
| | Reported | 0.90 | 0.30 | 0.36 | 1.52 | 1.17** | 0.46 | -0.08 | 0.23 |
| | | logML | | | | $CSR^2$ | | $CSR^2$ | $CSR^2_{res}$ |
| | | 1779 | | | | 94.01 | | 6.78 | 85.36 |

(ii) the three-year measure from Parker and Julliard (2005) ("P-J"), (iii) the fourth-quarter-to-fourth-quarter measure of Jagannathan and Wang (2007) ("Q4-Q4"), (iv) the garbage-based measure from Savov (2011) ("Garbage"), and (v) the unfiltered NIPA measure from Kroencke (2017) ("Unfiltered"). Details are provided in Appendix IA.I.2. The test assets consist of 30 portfolios sorted by size, value, and investment, along with 5 nontraded consumption-based factors. We follow the multifactor specification in Kleibergen and Zhan (2020) and treat MktRf as a traded factor.[9] As the data are annual, the first 12 years constitute the training sample. Table 11 presents the single-factor comparison with MktRf.

The single-factor CAPM yields the highest ML, significantly surpassing all two-factor models that include MktRf and an additional consumption measure. The posterior model probability based on ML ratios suggests that the probability of CAPM, excluding consumption factors, being the optimal model approaches one. This finding from our Bayesian model comparison is not surprising and aligns with Kleibergen and Zhan (2020), which employs an $F$ identification test to demonstrate the weakness of all consumption measures. They fail to reject the null hypothesis that $\boldsymbol{B} = 0$ for all five consumption measures. However, the conclusion differs based on CSR$^2$s, which indicates the Rank 5 model as optimal in Table 11.

In summary, employing our Bayesian model comparison framework, we reexamine key theory-based consumption factors, addressing the challenges of weak factor identification. Our analysis corroborates Kleibergen and Zhan (2020), reaffirming that incorporating multiple consumption growth measures does not significantly enhance trade factor models, such as the single-factor CAPM.

## 6 Summary

Beta-pricing models integrate asset-specific return dynamics and cross-sectional pricing restrictions, offering a joint framework for understanding returns and average returns. Yet, significant empirical challenges remain. First, standard estimation methods fail to account for uncertainty in parameters and models. Second, traditional metrics, such as cross-sectional $R^2$ and Hansen-Jagannathan distance, are limited in model comparison, as larger models perform at least as well as smaller ones. Third, weak factors have long challenged model estimation and inference.

We propose a Bayesian framework for estimating beta-pricing models with traded

[9]The other specification in Kleibergen and Zhan (2020) treats MktRf as a test asset rather than a factor. However, the limitations of consumption measures remain unexamined without a traded factor benchmark.

and nontraded factors, incorporating asset pricing restrictions following Lewellen, Nagel, and Shanken (2010). This framework enables model comparison via marginal likelihoods, excluding weak factors based on overall fit. Simulations show that the marginal likelihood criterion balances fit and complexity, reducing bias toward larger models relative to traditional metrics. Furthermore, our model provides two novel insights on factor evaluation derived from the marginal likelihood: (i) a factor with zero risk premia can still enhance model fit if it has large betas; (ii) a factor with nonzero risk premia may not enhance marginal likelihood if it is weak.

Empirically, we test each nontraded factor and assess its incremental contribution relative to benchmark traded factor models. Among 18 candidate factors (10 traded, 8 nontraded), the optimal model selects 8 traded factors, excluding all nontraded ones. These results suggest that all tested nontraded factors are weak. Furthermore, the selected model delivers superior out-of-sample tangency portfolio performance. Revisiting Kan, Robotti, and Shanken (2013), our method identifies the Fama-French three-factor model as optimal, excluding all nontraded factors. In evaluating consumption-based models from Kleibergen and Zhan (2020), all consumption measures appear weak and are excluded when controlling for the market factor.

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# Appendix

## A.I MCMC and Marginal Likelihood: General Case

**(MCMC Algorithm)** Initialize parameters and iterate

1. **Sample** $\boldsymbol{\beta}_i$ from

$$\boldsymbol{\beta}_i \mid \text{rest} \sim \mathcal{N}(\widehat{\boldsymbol{\beta}}_i, \mathcal{B}_i)$$

where

$$\widehat{\boldsymbol{\beta}}_i = \mathcal{B}_{0i}\boldsymbol{\beta}_{0i} + \tilde{\boldsymbol{F}}'\boldsymbol{R}_i/\sigma_i^2$$
$$\mathcal{B}_i^{-1} = \mathcal{B}_{0i}^{-1} + \tilde{\boldsymbol{F}}'\tilde{\boldsymbol{F}}/\sigma_i^2$$
$$\tilde{\boldsymbol{F}}' = (\tilde{\boldsymbol{f}}_1, ..., \tilde{\boldsymbol{f}}_t)'$$

2. **Sample** $\sigma_{ei}^2$ from

$$\sigma_{ei}^2 \mid \text{rest} \sim \mathcal{IG}\left(\nu_{ei}, \delta_{ei}\right)$$

where

$$\nu_{ei} = \frac{\nu_{e,0i} + T}{2}$$
$$\delta_e = \frac{\delta_{e,0i} + SSE_i}{2}$$
$$SSE_i = \text{sum}(\boldsymbol{E}_i \odot \boldsymbol{E}_i)$$
$$\boldsymbol{E}_i = \boldsymbol{R}_i - \tilde{\boldsymbol{F}}\boldsymbol{\beta}_i'$$

3. **Sample** $\boldsymbol{\lambda}^{NT}$ from

$$\boldsymbol{\lambda}^{NT} \mid \text{rest} \sim \mathcal{N}(\widehat{\boldsymbol{\lambda}}^{NT}, \boldsymbol{L}^{NT})$$

where

$$\left(\boldsymbol{L}^{NT}\right)^{-1} = \left(\boldsymbol{L}_0^{NT}\right)^{-1} + T\boldsymbol{B}^{NT\prime}\boldsymbol{\Sigma}_e^{-1}\boldsymbol{B}^{NT}$$
$$\widehat{\boldsymbol{\lambda}}^{NT} = \boldsymbol{L}^{NT}\left(\left(\boldsymbol{L}_0^{NT}\right)^{-1}\boldsymbol{\lambda}_0^{NT} + \boldsymbol{B}^{NT\prime}\boldsymbol{\Sigma}_e^{-1}(\boldsymbol{R} - \tilde{\boldsymbol{F}}\boldsymbol{B}')\boldsymbol{1}_T\right)$$

4. **Sample** $\boldsymbol{\mu}$ from

$$\boldsymbol{\mu} \sim \mathcal{N}(\widehat{\boldsymbol{\mu}}, \boldsymbol{D}_1)$$

where

$$\widehat{\boldsymbol{\mu}} = \boldsymbol{D}_1(\boldsymbol{M}_1^{-1}\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_u^{-1}\boldsymbol{F}'\boldsymbol{1}_T)$$
$$\boldsymbol{D}_1^{-1} = \boldsymbol{M}_1^{-1} + T\boldsymbol{\Sigma}_u^{-1}$$

and

$$\boldsymbol{\mu}_1 = \boldsymbol{M}_1(\boldsymbol{M}_0^{-1}\boldsymbol{\mu}_0 + \tilde{\boldsymbol{B}}'\boldsymbol{\Sigma}_e^{-1}\boldsymbol{R}_*)$$
$$\boldsymbol{M}_1^{-1} = \boldsymbol{M}_0^{-1} + T\tilde{\boldsymbol{B}}'\boldsymbol{\Sigma}_e^{-1}\tilde{\boldsymbol{B}}$$
$$\boldsymbol{R}_* = colsum(\boldsymbol{R} - \boldsymbol{F}\boldsymbol{B}' - \mathbf{1}'\boldsymbol{\lambda}^{NT'}\boldsymbol{B}^{NT'})'$$

$\tilde{\boldsymbol{B}}$ is an $n \times k^*$ matrix where the corresponding $\boldsymbol{f}^{NT}$ columns contain the $\boldsymbol{\beta}$ for $\boldsymbol{f}^{NT}$ and leave the other columns zero.

5. **Sample** $\boldsymbol{\Sigma}_u$ from

$$\boldsymbol{\Sigma}_u \mid \text{rest} \sim \mathcal{IW}(\nu_u, \boldsymbol{S}_u)$$

where

$$\nu_u = \rho_{u,0} + T$$
$$\boldsymbol{S}_u = \boldsymbol{Q}_{u,0} + \boldsymbol{Q}$$
$$\boldsymbol{Q} = (\boldsymbol{F} - \mathbf{1}_T\boldsymbol{\mu}')'(\boldsymbol{F} - \mathbf{1}_T\boldsymbol{\mu}')$$

**(Marginal Likelihood)** To estimate the marginal likelihood of each contending model, we employ the Chib (1995) method, which starts with the convenient expression of the log-marginal likelihood

$$\ln m(\boldsymbol{y}_{1:T} \mid \mathcal{M}_j) = \ln \pi(\boldsymbol{\theta}^* \mid \mathcal{M}_j) + \ln p(\boldsymbol{y}_{1:T} \mid \mathcal{M}_j, \boldsymbol{\theta}^*)$$
$$- \ln \pi(\boldsymbol{\theta}^* \mid \mathcal{M}_j, \boldsymbol{y}_{1:T})$$

where $\boldsymbol{\theta}^* = (\boldsymbol{B}^*, \boldsymbol{\Sigma}_e^*, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_u^*, \boldsymbol{\mu}^*)$ is some chosen point, say the posterior mean. In this expression, the prior and likelihood ordinates can be found analytically. As for the third term, the posterior ordinate suppresses the model index and uses a marginal-conditional decomposition to write

$$\ln \pi(\boldsymbol{\theta}^* \mid \boldsymbol{y}_{1:T}) = \ln \pi(\boldsymbol{B}^* \mid \boldsymbol{y}_{1:T}) + \ln \pi(\boldsymbol{\Sigma}_e^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{B}^*)$$
$$+ \ln \pi(\boldsymbol{\lambda}^{NT*} \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^*) + \ln \pi(\boldsymbol{\Sigma}_u^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^*)$$
$$+ \ln \pi(\boldsymbol{\mu}^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_u^*, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^*)$$

An estimate of the first ordinate is obtained as

$$\pi(\boldsymbol{B}^* \mid \boldsymbol{y}_{1:T}) = \frac{1}{G}\sum_{g=1}^{G}\prod_{i=1}^{n}\mathcal{N}\left(\mathcal{B}_{0i}\beta_{0i} + \tilde{\boldsymbol{F}}'^{(g)}\boldsymbol{R}_i/\sigma_i^{2(g)}, (\mathcal{B}_{0i}^{-1} + \tilde{\boldsymbol{F}}'^{(g)}\tilde{\boldsymbol{F}}^{(g)}/\sigma_i^{2(g)})^{-1}\right)$$

the second from reduced run 1

$$\pi(\boldsymbol{\Sigma}_e^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{B}^*) = \frac{1}{G}\sum_{g=1}^{G}\prod_{i=1}^{n}\mathcal{IG}\left(\frac{\nu_{e,0i}+T}{2}, \frac{\delta_{e,0i} + \mathrm{sum}((\boldsymbol{R}_i - \tilde{\boldsymbol{F}}^{(g)}\boldsymbol{\beta}_i^{*\prime}) \odot (\boldsymbol{R}_i - \tilde{\boldsymbol{F}}^{(g)}\boldsymbol{\beta}_i^{*\prime}))}{2}\right)$$

the third from reduced run 2

$$\pi(\boldsymbol{\lambda}^{NT*} \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^*) = \frac{1}{G}\sum_{g=1}^{G}\mathcal{N}\Bigg(\boldsymbol{L}^{NT}\Big((\boldsymbol{L}_0^{NT})^{-1}\boldsymbol{\lambda}_0^{NT} + \boldsymbol{B}^{*\prime NT}\boldsymbol{\Sigma}_e^{*-1}(\boldsymbol{R} - \tilde{\boldsymbol{F}}^{(g)}\boldsymbol{B}^{*\prime})\mathbf{1}_T\Big), \left((\boldsymbol{L}_0^{NT})^{-1} + T\boldsymbol{B}^{*\prime NT}\boldsymbol{\Sigma}_e^{*-1}\boldsymbol{B}^{*NT}\right)^{-1}\Bigg)$$

the fourth from reduced run 3

$$\pi(\boldsymbol{\Sigma}_u^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^*) = \mathcal{IW}\left(\rho_{u,0} + T, \boldsymbol{Q}_{u,0} + (\boldsymbol{F} - \mathbf{1}_T\boldsymbol{\mu}^{(g)\prime})'(\boldsymbol{F} - \mathbf{1}_T\boldsymbol{\mu}^{(g)\prime})\right)$$

finally, the fifth from the chosen point

$$\pi(\boldsymbol{\mu}^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_u^*, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^*) = \mathcal{N}\left(\widehat{\boldsymbol{\mu}}^*, \boldsymbol{D}_1^*\right)$$

where

$$\widehat{\boldsymbol{\mu}}^* = \boldsymbol{D}_1(\boldsymbol{M}_1^{*-1}\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_u^{*-1}\boldsymbol{F}'\mathbf{1}_T)$$
$$\boldsymbol{D}_1^{-1} = \boldsymbol{M}_1^{*-1} + T\boldsymbol{\Sigma}_u^{*-1}$$

and

$$\boldsymbol{\mu}_1 = \boldsymbol{M}_1^*(\boldsymbol{M}_0^{-1}\boldsymbol{\mu}_0 + \tilde{\boldsymbol{B}}^{*\prime}\boldsymbol{\Sigma}_e^{*-1}\boldsymbol{R}_*)$$
$$\boldsymbol{M}_1^{*-1} = \boldsymbol{M}_0^{-1} + T\tilde{\boldsymbol{B}}^{*\prime}\boldsymbol{\Sigma}_e^{*-1}\tilde{\boldsymbol{B}}^*$$
$$\boldsymbol{R}_* = colsum(\boldsymbol{R} - \boldsymbol{F}\boldsymbol{B}^{*\prime} - \mathbf{1}'\boldsymbol{\lambda}^{NT*\prime}\boldsymbol{B}^{*NT\prime})'$$

## A.II MCMC and Marginal Likelihood: NT Factors Only

In this section, we propose the model specification with only nontraded factors, such as a consumption-based model. Start with the restricted TS model, which is the same as equation 11

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t \tag{A.1}$$

and factors

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t \tag{A.2}$$

The observed data is $\boldsymbol{y}_{1:T} = \{\boldsymbol{r}_{1:T}, \boldsymbol{f}_{1:T}^*\}$, excluding the training sample. We first specify the conditional models used to construct the posterior distributions involved in the

MCMC sampling. For updating $\boldsymbol{B}^{NT} = [\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_n]$ and $\boldsymbol{\Sigma}_e$ we use the model

$$\boldsymbol{r}_t = \boldsymbol{B}^{NT} \tilde{\boldsymbol{f}}_t^{NT} + \boldsymbol{e}_t$$

where

$$\tilde{\boldsymbol{f}}_t^{NT} = \boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT} + \boldsymbol{\lambda}^{NT} \tag{A.3}$$

For updating $\boldsymbol{\Sigma}_u$ we use the model

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t$$

For updating $\boldsymbol{\lambda}^{NT}$ we use the model

$$\boldsymbol{r}_t - \boldsymbol{B}^{NT} \boldsymbol{f}_t^{NT} + \boldsymbol{B}^{NT} \boldsymbol{\mu}^{NT} = \boldsymbol{\lambda}^{NT} + \boldsymbol{e}_t$$

Finally, for updating $\boldsymbol{\mu}$ we use the models

$$\boldsymbol{r}_t - \boldsymbol{B}^{NT} \boldsymbol{\lambda}^{NT} - \boldsymbol{B}^{NT} \boldsymbol{f}_t^{NT} = -\boldsymbol{B}^{NT} \boldsymbol{\mu}^{NT} + \boldsymbol{e}_t$$

and

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t$$

**(MCMC Algorithm: NT Factors Only)** Initialize parameters and iterate

1. **Sample** $\boldsymbol{\beta}_i$ from

   $$\boldsymbol{\beta}_i \mid \text{rest} \sim \mathcal{N}(\widehat{\boldsymbol{\beta}}_i, \mathcal{B}_i)$$

   where

   $$\widehat{\boldsymbol{\beta}}_i = \mathcal{B}_{0i} \boldsymbol{\beta}_{0i} + \tilde{\boldsymbol{F}}' R_i / \sigma_i^2$$
   $$\mathcal{B}_i^{-1} = \mathcal{B}_{0i}^{-1} + \tilde{\boldsymbol{F}}' \tilde{\boldsymbol{F}} / \sigma_i^2$$
   $$\tilde{\boldsymbol{F}}' = (\tilde{\boldsymbol{f}}_1^{NT}, ..., \tilde{\boldsymbol{f}}_t^{NT})'$$

2. **Sample** $\sigma_{ei}^2$ from

   $$\sigma_{ei}^2 \mid \text{rest} \sim \mathcal{IG}\left(\nu_{ei}, \delta_{ei}\right)$$

   where

   $$\nu_{ei} = \frac{\nu_{e,0i} + T}{2}$$
   $$\delta_e = \frac{\delta_{e,0i} + SSE_i}{2}$$
   $$SSE_i = \text{sum}(\boldsymbol{E}_i \odot \boldsymbol{E}_i)$$
   $$\boldsymbol{E}_i = \boldsymbol{R}_i - \tilde{\boldsymbol{F}} \boldsymbol{\beta}_i'$$

3. **Sample** $\boldsymbol{\lambda}^{NT}$ from

   $$\boldsymbol{\lambda}^{NT} \mid \text{rest} \sim \mathcal{N}(\widehat{\boldsymbol{\lambda}^{NT}}, \boldsymbol{L})$$

where

$$\boldsymbol{L}^{-1} = (\boldsymbol{L}_0)^{-1} + T\boldsymbol{B}'^{NT}\boldsymbol{\Sigma}_e^{-1}\boldsymbol{B}^{NT}$$
$$\widehat{\boldsymbol{\lambda}^{NT}} = \boldsymbol{L}\left((\boldsymbol{L}_0)^{-1}\boldsymbol{\lambda}_0 + \boldsymbol{B}'^{NT}\boldsymbol{\Sigma}_e^{-1}(\boldsymbol{R} - \tilde{\boldsymbol{F}}\boldsymbol{B}'^{NT})\mathbf{1}_T\right)$$

4. **Sample $\boldsymbol{\mu}$** from

$$\boldsymbol{\mu} \sim \mathcal{N}(\widehat{\boldsymbol{\mu}}, \boldsymbol{D}_1)$$

where

$$\widehat{\boldsymbol{\mu}} = \boldsymbol{D}_1(\boldsymbol{M}_1^{-1}\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_u^{-1}\boldsymbol{F}'\mathbf{1}_T)$$
$$\boldsymbol{D}_1^{-1} = \boldsymbol{M}_1^{-1} + T\boldsymbol{\Sigma}_u^{-1}$$

and

$$\boldsymbol{\mu}_1 = \boldsymbol{M}_1(\boldsymbol{M}_0^{-1}\boldsymbol{\mu}_0 + \tilde{\boldsymbol{B}}'\boldsymbol{\Sigma}_e^{-1}\boldsymbol{R}_*)$$
$$\boldsymbol{M}_1^{-1} = \boldsymbol{M}_0^{-1} + T\tilde{\boldsymbol{B}}'\boldsymbol{\Sigma}_e^{-1}\tilde{\boldsymbol{B}}$$
$$\boldsymbol{R}_* = colsum(\boldsymbol{R} - \boldsymbol{F}^{NT}\boldsymbol{B}'^{NT} - \mathbf{1}'\boldsymbol{\lambda}'^{NT}\boldsymbol{B}'^{NT})'$$

$\tilde{\boldsymbol{B}}$ is an $n \times k^*$ matrix where the corresponding $f$ columns contain the $\boldsymbol{\beta}$ for $f$ and leave the other columns zero.

5. **Sample $\Sigma_u$** from

$$\boldsymbol{\Sigma}_u \mid \text{rest} \sim \mathcal{IW}(\nu_u, \boldsymbol{S}_u)$$

where

$$\nu_u = \rho_{u,0} + T$$
$$\boldsymbol{S}_u = \boldsymbol{Q}_{u,0} + \boldsymbol{Q}$$
$$\boldsymbol{Q} = (\boldsymbol{F} - \mathbf{1}_T\boldsymbol{\mu}')'(\boldsymbol{F} - \mathbf{1}_T\boldsymbol{\mu}')$$

The marginal likelihood estimation is almost the same as that of both traded and nontraded factors. Still, the only difference is that $\tilde{\boldsymbol{F}}$ is not a combination of traded and nontraded factors, but the adjusted nontraded factor in equation A.3.

**(Marginal Likelihood: NT Factors Only)** To estimate the marginal likelihood of each contending model, we again employ the Chib (1995) method, where now $\boldsymbol{\theta} = (\boldsymbol{B}^{NT}, \Sigma_e, \boldsymbol{\lambda}^{NT}, \Sigma_u, \boldsymbol{\mu})$. The posterior ordinate is obtained via the following marginal-conditional decomposition:

$$\begin{aligned}\ln \pi(\boldsymbol{\theta}^* \mid \boldsymbol{y}_{1:T}) = &\ln \pi(\boldsymbol{B}^{*NT} \mid \boldsymbol{y}_{1:T}) + \ln \pi(\boldsymbol{\Sigma}_e^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{B}^{*NT}) \\ &+ \ln \pi(\boldsymbol{\lambda}^{NT*} \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*NT}) + \ln \pi(\boldsymbol{\Sigma}_u^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*NT}) \\ &+ \ln \pi(\boldsymbol{\mu}^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_u^*, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*NT}).\end{aligned}$$

The first ordinate is estimated as:

$$\begin{aligned}\pi(\boldsymbol{B}^{*NT} \mid \boldsymbol{y}_{1:T}) = \frac{1}{G}\sum_{g=1}^{G}\prod_{i=1}^{n}\mathcal{N}\Big(&\mathcal{B}_{0i}\boldsymbol{\beta}_{0i} + \tilde{\boldsymbol{F}}'^{(g)}R_i/\sigma_i^{2(g)}, \\ &(\mathcal{B}_{0i}^{-1} + \tilde{\boldsymbol{F}}'^{(g)}\tilde{\boldsymbol{F}}^{(g)}/\sigma_i^{2(g)})^{-1}\Big).\end{aligned}$$

The second is estimated from reduced run 1:

$$\begin{aligned}\pi(\boldsymbol{\Sigma}_e^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{B}^{*NT}) = \frac{1}{G}\sum_{g=1}^{G}\prod_{i=1}^{n}\mathcal{IG}\Bigg(&\frac{\nu_{e,0i}+T}{2}, \\ &\frac{\delta_{e,0i} + \mathrm{sum}((\boldsymbol{R}_i - \tilde{\boldsymbol{F}}^{(g)}\boldsymbol{\beta}_i^{*\prime}) \odot (\boldsymbol{R}_i - \tilde{\boldsymbol{F}}^{(g)}\boldsymbol{\beta}_i^{*\prime}))}{2}\Bigg).\end{aligned}$$

The third from reduced run 2:

$$\begin{aligned}\pi(\boldsymbol{\lambda}^{NT*} \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*NT}) = \frac{1}{G}\sum_{g=1}^{G}\mathcal{N}\Bigg(&\boldsymbol{L}^{NT}\Big((\boldsymbol{L}_0^{NT})^{-1}\boldsymbol{\lambda}_0^{NT} + \boldsymbol{B}^{*'NT}\boldsymbol{\Sigma}_e^{*-1} \\ &\cdot(\boldsymbol{R} - \tilde{\boldsymbol{F}}^{(g)}\boldsymbol{B}^{*'NT})\boldsymbol{1}_T\Big), \Big((\boldsymbol{L}_0^{NT})^{-1} + T\boldsymbol{B}^{*'NT}\boldsymbol{\Sigma}_e^{*-1}\boldsymbol{B}^{*NT}\Big)^{-1}\Bigg).\end{aligned}$$

The fourth from reduced run 3:

$$\begin{aligned}\pi(\boldsymbol{\Sigma}_u^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*NT}) = \mathcal{IW}\big(&\rho_{u,0} + T, \\ &\boldsymbol{Q}_{u,0} + (\boldsymbol{F} - \boldsymbol{1}_T\boldsymbol{\mu}^{(g)'})'(\boldsymbol{F} - \boldsymbol{1}_T\boldsymbol{\mu}^{(g)'})\big).\end{aligned}$$

Finally, the fifth is from the chosen point:

$$\pi(\boldsymbol{\mu}^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_u^*, \boldsymbol{\lambda}^{NT*}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*NT}) = \mathcal{N}\left(\widehat{\boldsymbol{\mu}}^*, \boldsymbol{D}_1^*\right),$$

where:

$$\widehat{\boldsymbol{\mu}}^* = \boldsymbol{D}_1(\boldsymbol{M}_1^{*-1}\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_u^{*-1}\boldsymbol{F}'\boldsymbol{1}_T), \quad \boldsymbol{D}_1^{-1} = \boldsymbol{M}_1^{*-1} + T\boldsymbol{\Sigma}_u^{*-1}.$$

and

$$\boldsymbol{\mu}_1 = \boldsymbol{M}_1^*(\boldsymbol{M}_0^{-1}\boldsymbol{\mu}_0 + \tilde{\boldsymbol{B}}^{*'}\boldsymbol{\Sigma}_e^{*-1}\boldsymbol{R}_*), \quad \boldsymbol{M}_1^{*-1} = \boldsymbol{M}_0^{-1} + T\tilde{\boldsymbol{B}}^{*'}\boldsymbol{\Sigma}_e^{*-1}\tilde{\boldsymbol{B}}^*.$$

Finally, we define:

$$\boldsymbol{R}_* = \mathrm{colsum}(\boldsymbol{R} - \boldsymbol{F}\boldsymbol{B}^{*NT'} - \boldsymbol{1}'\boldsymbol{\lambda}^{NT*'}\boldsymbol{B}^{*NT'})'.$$

## A.III MCMC and Marginal Likelihood: T Factors Only

In this section, we propose the model specification with only traded factors, for example, the Fama-French 3-factor model. In this case, the restricted TS model takes a different form without $\lambda$

$$\boldsymbol{r}_t = \boldsymbol{B}^T \boldsymbol{f}_t^T + \boldsymbol{e}_t \tag{A.4}$$

and factors

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t \tag{A.5}$$

The observed data is $\boldsymbol{y}_{1:T} = \{r_{1:T}, f_{1:T}^*\}$ denotes the sample data exclude the training sample. We first specify the conditional model used to sample these posterior distributions. For updating $\boldsymbol{B}^T = [\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_n]$ and $\boldsymbol{\Sigma}_e$ we use the model

$$\boldsymbol{r}_t = \boldsymbol{B}^T \boldsymbol{f}_t^T + \boldsymbol{e}_t$$

For updating $\boldsymbol{\Sigma}_u$ we use the model

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t$$

Finally, for updating $\boldsymbol{\mu}$ we use the models

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t$$

**(MCMC Algorithm with Only Traded Factors)** Initialize parameters and iterate

1. **Sample** $\boldsymbol{\beta}_i$ from

$$\boldsymbol{\beta}_i \mid \text{rest} \sim \mathcal{N}(\widehat{\boldsymbol{\beta}}_i, \mathcal{B}_i)$$

where

$$\widehat{\boldsymbol{\beta}}_i = \mathcal{B}_{0i}\boldsymbol{\beta}_{0i} + \boldsymbol{F}' R_i/\sigma_i^2$$
$$\mathcal{B}_i^{-1} = \mathcal{B}_{0i}^{-1} + \boldsymbol{F}'\boldsymbol{F}/\sigma_i^2$$
$$\boldsymbol{F}' = (\boldsymbol{f}_1^T, ..., \boldsymbol{f}_t^T)'$$

2. **Sample** $\sigma_{ei}^2$ from

$$\sigma_{ei}^2 \mid \text{rest} \sim \mathcal{IG}\left(\nu_{ei}, \delta_{ei}\right)$$

where

$$\nu_{ei} = \frac{\nu_{e,0i} + T}{2}$$
$$\delta_e = \frac{\delta_{e,0i} + SSE_i}{2}$$
$$SSE_i = \text{sum}(\boldsymbol{E}_i \odot \boldsymbol{E}_i)$$

$$\boldsymbol{E}_i = \boldsymbol{R}_i - \boldsymbol{F}\boldsymbol{\beta}_i'$$

3. **Sample $\boldsymbol{\mu}$** from

$$\boldsymbol{\mu} \sim \mathcal{N}(\widehat{\boldsymbol{\mu}}, \boldsymbol{D}_1)$$

where

$$\widehat{\boldsymbol{\mu}} = \boldsymbol{D}_1(\boldsymbol{M}_0^{-1}\boldsymbol{\mu}_0 + \boldsymbol{\Sigma}_u^{-1}\boldsymbol{F}'\boldsymbol{1}_T)$$
$$\boldsymbol{D}_1^{-1} = \boldsymbol{M}_0^{-1} + T\boldsymbol{\Sigma}_u^{-1}$$

4. **Sample $\Sigma_u$** from

$$\boldsymbol{\Sigma}_u \mid \text{rest} \sim \mathcal{IW}\left(\nu_u, \boldsymbol{S}_u\right)$$

where

$$\nu_u = \rho_{u,0} + T$$
$$S_u = \boldsymbol{Q}_{u,0} + \boldsymbol{Q}$$
$$\boldsymbol{Q} = (\boldsymbol{F} - \boldsymbol{1}_T\boldsymbol{\mu}')'(\boldsymbol{F} - \boldsymbol{1}_T\boldsymbol{\mu}')$$

**(Marginal Likelihood: T Factors Only)** To estimate the marginal likelihood of each contending model, we again employ the Chib (1995) method where now $\boldsymbol{\theta} = (\boldsymbol{B}^T, \Sigma_e, \Sigma_u, \boldsymbol{\mu})$. The posterior ordinate is obtained via the following marginal-conditional decomposition:

$$\ln \pi(\boldsymbol{\theta}^* \mid \boldsymbol{y}_{1:T}) = \ln \pi(\boldsymbol{B}^{*T} \mid \boldsymbol{y}_{1:T}) + \ln \pi(\boldsymbol{\Sigma}_e^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{B}^{*T})$$
$$+ \ln \pi(\boldsymbol{\Sigma}_u^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*T}) + \ln \pi(\boldsymbol{\mu}^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_u^*, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*T})$$

An estimate of the first ordinate is obtained as

$$\pi(\boldsymbol{B}^{*T} \mid \boldsymbol{y}_{1:T}) = \frac{1}{G}\sum_{g=1}^{G}\prod_{i=1}^{n}\mathcal{N}\Big(\mathcal{B}_{0i}\boldsymbol{\beta}_{0i} + \boldsymbol{F}'^{(g)}R_i/\sigma_i^{2(g)},$$
$$(\mathcal{B}_{0i}^{-1} + \boldsymbol{F}'^{(g)}\boldsymbol{F}^{(g)}/\sigma_i^{2(g)})^{-1}\Big)$$

the second from reduced run 1

$$\pi(\boldsymbol{\Sigma}_e^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{B}^{*T}) = \frac{1}{G}\sum_{g=1}^{G}\prod_{i=1}^{n}\mathcal{IG}\left(\frac{\nu_{e,0i} + T}{2},\right.$$
$$\left.\frac{\delta_{e,0i} + \text{sum}((\boldsymbol{R}_i - \boldsymbol{F}^{(g)}\boldsymbol{\beta}_i^{*\prime}) \odot (\boldsymbol{R}_i - \boldsymbol{F}^{(g)}\boldsymbol{\beta}_i^{*\prime}))}{2}\right)$$

the third from reduced run 2

$$\pi(\boldsymbol{\Sigma}_u^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*T}) = \mathcal{IW}\Big(\rho_{u,0} + T,$$
$$\boldsymbol{Q}_{u,0} + (\boldsymbol{F} - \boldsymbol{1}_T\boldsymbol{\mu}^{(g)\prime})'(\boldsymbol{F} - \boldsymbol{1}_T\boldsymbol{\mu}^{(g)\prime})\Big)$$

finally, the fourth from the chosen point

$$\pi(\boldsymbol{\mu}^* \mid \boldsymbol{y}_{1:T}, \boldsymbol{\Sigma}_u^*, \boldsymbol{\Sigma}_e^*, \boldsymbol{B}^{*T}) = \mathcal{N}\left(\widehat{\boldsymbol{\mu}}^*, \boldsymbol{D}_1^*\right)$$

where

$$\widehat{\boldsymbol{\mu}}^* = \boldsymbol{D}_1(\boldsymbol{M}_0^{*-1}\boldsymbol{\mu}_0 + \boldsymbol{\Sigma}_u^{*-1}\boldsymbol{F}'\mathbf{1}_T)$$
$$\boldsymbol{D}_1^{-1} = \boldsymbol{M}_0^{*-1} + T\boldsymbol{\Sigma}_u^{*-1}$$

# Internet Appendix

(Not for Publication)

## IA.I Data Description

### IA.I.1 Data in Kan, Robotti, and Shanken (2013)

There are 8 asset pricing models with 14 factors in Kan et al. (2013) that are directly downloaded from Prof. Cesare Robotti's website, including

1. *vw*, *smb*, *hml*, directly downloaded from French's website.

2. *prem*, the lagged yield spread between Baa- and Aaa-rated corporate bonds.

3. *lab*, the growth rate of per capita labor income, $L$, is the difference between total personal income and dividend payments, scaled by the total population.

4. *term*, the yield spread between 10-year and 1-year Treasury bonds. *def*, the credit spread between Baa-rated corporate bonds and long-term Treasury bonds. *div*, the dividend yield. *rf*, the 1-month T-bill yield. The factors *term*, *def*, *div*, and *rf* are extracted as innovations from a VAR(1) model with seven state variables, including *vw*, *smb*, and *hml*.

5. *cg* represents the growth rate in real per capita nondurable consumption (seasonally adjusted at annual rates).

6. *cay*, the conditioning variable, represents the consumption-aggregate wealth ratio as defined by Lettau and Ludvigson (2001). Quarterly *cay* values are linearly interpolated to enable monthly frequency analysis. The variable cgcay is the product of *cg* and *cay*.

7. *cg36*, the growth rate in real per capita nondurable consumption over 3 years starting with the given month.

8. *cgdur*, the growth rate in real per capita durable consumption (seasonally adjusted at annual rates).

Models that are estimated:

1. CAPM

$$E(\boldsymbol{r}_t) = \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T} \underbrace{[\lambda_{vw}]}_{\lambda^T}$$

$$\boldsymbol{r}_t = \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T} \underbrace{[f_{vw}]}_{f^T} + \boldsymbol{e}_t$$

2. The conditional CAPM (C-LAB) of Jagannathan and Wang (1996)

$$E(\boldsymbol{r}_t) = \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T} \underbrace{[\lambda_{vw}]}_{\lambda^T} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{lab} & \boldsymbol{\beta}_{\text{prem}}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{lab} \\ \lambda_{\text{prem}}\end{bmatrix}}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{lab} & \boldsymbol{\beta}_{\text{prem}}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{lab} \\ \lambda_{\text{prem}}\end{bmatrix}}_{\lambda^{NT}} + \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T} \underbrace{[f_{vw}]}_{f^T} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{lab} & \boldsymbol{\beta}_{\text{prem}}\end{bmatrix}}_{B^{NT}} \left( \underbrace{\begin{bmatrix}f_{lab} \\ f_{\text{prem}}\end{bmatrix}}_{f^{NT}} - \underbrace{\begin{bmatrix}\mu_{lab} \\ \mu_{\text{prem}}\end{bmatrix}}_{\mu^{NT}} \right) + \boldsymbol{e}_t$$

3. FF3

$$E(\boldsymbol{r}_t) = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw} & \boldsymbol{\beta}_{smb} & \boldsymbol{\beta}_{hml}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}\lambda_{vw} \\ \lambda_{smb} \\ \lambda_{hml}\end{bmatrix}}_{\lambda^T}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw} & \boldsymbol{\beta}_{smb} & \boldsymbol{\beta}_{hml}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}f_{vw} \\ f_{smb} \\ f_{hml}\end{bmatrix}}_{f^T} + \boldsymbol{e}_t$$

4. ICAPM (Petkova, 2006)

$$E(\boldsymbol{r}_t) = \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T} \underbrace{[\lambda_{vw}]}_{\lambda^T} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{term} & \boldsymbol{\beta}_{def} & \boldsymbol{\beta}_{div} & \boldsymbol{\beta}_{rf}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{term} \\ \lambda_{def} \\ \lambda_{div} \\ \lambda_{rf}\end{bmatrix}}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{term} & \boldsymbol{\beta}_{def} & \boldsymbol{\beta}_{div} & \boldsymbol{\beta}_{rf}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{term} \\ \lambda_{def} \\ \lambda_{div} \\ \lambda_{rf}\end{bmatrix}}_{\lambda^{NT}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^{T}} \underbrace{\begin{bmatrix}f_{vw}\end{bmatrix}}_{f^{T}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{term} & \boldsymbol{\beta}_{def} & \boldsymbol{\beta}_{div} & \boldsymbol{\beta}_{rf}\end{bmatrix}}_{B^{NT}} \left( \underbrace{\begin{bmatrix}f_{term} \\ f_{def} \\ f_{div} \\ f_{rf}\end{bmatrix}}_{f^{NT}} - \underbrace{\begin{bmatrix}\mu_{term} \\ \mu_{def} \\ \mu_{div} \\ \mu_{rf}\end{bmatrix}}_{\mu^{NT}} \right) + \boldsymbol{e}_t$$

5. The unconditional consumption model (CCAPM)

$$E(\boldsymbol{r}_t) = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{cg}\end{bmatrix}}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{cg}\end{bmatrix}}_{\lambda^{NT}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg}\end{bmatrix}}_{B^{NT}} \left( \underbrace{\begin{bmatrix}f_{cg}\end{bmatrix}}_{f^{NT}} - \underbrace{\begin{bmatrix}\mu_{cg}\end{bmatrix}}_{\mu^{NT}} \right) + \boldsymbol{e}_t$$

6. CC-CAY from Lettau and Ludvigson (2001)

$$E(\boldsymbol{r}_t) = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cay} & \boldsymbol{\beta}_{cg} & \boldsymbol{\beta}_{cg\cdot cay}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{cay} \\ \lambda_{cg} \\ \lambda_{cg\cdot cay}\end{bmatrix}}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cay} & \boldsymbol{\beta}_{cg} & \boldsymbol{\beta}_{cg\cdot cay}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{cay} \\ \lambda_{cg} \\ \lambda_{cg\cdot cay}\end{bmatrix}}_{\lambda^{NT}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cay} & \boldsymbol{\beta}_{cg} & \boldsymbol{\beta}_{cg\cdot cay}\end{bmatrix}}_{B^{NT}} \left( \underbrace{\begin{bmatrix}f_{cay} \\ f_{cg} \\ f_{cg\cdot cay}\end{bmatrix}}_{f^{NT}} - \underbrace{\begin{bmatrix}\mu_{cay} \\ \mu_{cg} \\ \mu_{cg\cdot cay}\end{bmatrix}}_{\mu^{NT}} \right) + \boldsymbol{e}_t$$

7. U-CCAPM from Parker and Julliard (2005)

$$E(\boldsymbol{r}_t) = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg36}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{cg36}\end{bmatrix}}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg36}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{cg36}\end{bmatrix}}_{\lambda^{NT}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg36}\end{bmatrix}}_{B^{NT}} \left( \underbrace{\begin{bmatrix}f_{cg36}\end{bmatrix}}_{f^{NT}} - \underbrace{\begin{bmatrix}\mu_{cg36}\end{bmatrix}}_{\mu^{NT}} \right) + \boldsymbol{e}_t$$

8. D-CCAPM from Yogo (2006)

$$E(\boldsymbol{r}_t) = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^{T}} \underbrace{\begin{bmatrix}\lambda_{vw}\end{bmatrix}}_{\lambda^{T}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg} & \boldsymbol{\beta}_{cgdur}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{cg} \\ \lambda_{cgdur}\end{bmatrix}}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg} & \boldsymbol{\beta}_{cgdur}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{cg} \\ \lambda_{cgdur}\end{bmatrix}}_{\lambda^{NT}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}f_{vw}\end{bmatrix}}_{f^T} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{cg} & \boldsymbol{\beta}_{cgdur}\end{bmatrix}}_{B^{NT}} \left( \underbrace{\begin{bmatrix}f_{cg} \\ f_{cgdur}\end{bmatrix}}_{f^{NT}} - \underbrace{\begin{bmatrix}\mu_{cg} \\ \mu_{cgdur}\end{bmatrix}}_{\mu^{NT}} \right) + \boldsymbol{e}_t$$

### IA.I.2 Data in Kleibergen and Zhan (2020)

Five consumption measures are utilized in Kleibergen and Zhan (2020). Thanks to Kroencke (2017), four of these measures are publicly available. Garbage data is sourced from the U.S. Environmental Protection Agency (EPA). Following Savov (2011), yard trimmings are excluded due to measurement issues. Instead of the commonly used Fama-French portfolios, univariate portfolios sorted by size, value, and investment are employed as test assets, which can also be accessed via Prof. Tim A. Kroencke's website.

Models that are estimated:

1. CAPM

$$E(\boldsymbol{r}_t) = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}\lambda_{vw}\end{bmatrix}}_{\lambda^T}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}f_{vw}\end{bmatrix}}_{f^T} + \boldsymbol{e}_t$$

2. MktRf + Reported

$$E(\boldsymbol{r}_t) = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}\lambda_{vw}\end{bmatrix}}_{\lambda^T} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{Reported}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{Reported}\end{bmatrix}}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{Reported}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{Reported}\end{bmatrix}}_{\lambda^{NT}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}f_{vw}\end{bmatrix}}_{f^T} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{Reported}\end{bmatrix}}_{B^{NT}} \left( \underbrace{\begin{bmatrix}f_{Reported}\end{bmatrix}}_{f^{NT}} - \underbrace{\begin{bmatrix}\mu_{Reported}\end{bmatrix}}_{\mu^{NT}} \right) + \boldsymbol{e}_t$$

3. MktRf + P-J

$$E(\boldsymbol{r}_t) = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}\lambda_{vw}\end{bmatrix}}_{\lambda^T} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{P-J}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{P-J}\end{bmatrix}}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{P-J}\end{bmatrix}}_{B^{NT}} \underbrace{\begin{bmatrix}\lambda_{P-J}\end{bmatrix}}_{\lambda^{NT}} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{vw}\end{bmatrix}}_{B^T} \underbrace{\begin{bmatrix}f_{vw}\end{bmatrix}}_{f^T} + \underbrace{\begin{bmatrix}\boldsymbol{\beta}_{P-J}\end{bmatrix}}_{B^{NT}} \left( \underbrace{\begin{bmatrix}f_{P-J}\end{bmatrix}}_{f^{NT}} - \underbrace{\begin{bmatrix}\mu_{P-J}\end{bmatrix}}_{\mu^{NT}} \right) + \boldsymbol{e}_t$$

4. MktRf + Q4-Q4

$$E(\boldsymbol{r}_t) = \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T}\underbrace{[\lambda_{vw}]}_{\lambda^T} + \underbrace{[\boldsymbol{\beta}_{Q4-Q4}]}_{B^{NT}}\underbrace{[\lambda_{Q4-Q4}]}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{[\boldsymbol{\beta}_{Q4-Q4}]}_{B^{NT}}\underbrace{[\lambda_{Q4-Q4}]}_{\lambda^{NT}} + \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T}\underbrace{[f_{vw}]}_{f^T} + \underbrace{[\boldsymbol{\beta}_{Q4-Q4}]}_{B^{NT}}\left(\underbrace{[f_{Q4-Q4}]}_{f^{NT}} - \underbrace{[\mu_{Q4-Q4}]}_{\mu^{NT}}\right) + \boldsymbol{e}_t$$

5. MktRf + Garbage

$$E(\boldsymbol{r}_t) = \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T}\underbrace{[\lambda_{vw}]}_{\lambda^T} + \underbrace{[\boldsymbol{\beta}_{Garbage}]}_{B^{NT}}\underbrace{[\lambda_{Garbage}]}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{[\boldsymbol{\beta}_{Garbage}]}_{B^{NT}}\underbrace{[\lambda_{Garbage}]}_{\lambda^{NT}} + \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T}\underbrace{[f_{vw}]}_{f^T} + \underbrace{[\boldsymbol{\beta}_{Garbage}]}_{B^{NT}}\left(\underbrace{[f_{Garbage}]}_{f^{NT}} - \underbrace{[\mu_{Garbage}]}_{\mu^{NT}}\right) + \boldsymbol{e}_t$$

6. MktRf + Unfiltered

$$E(\boldsymbol{r}_t) = \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T}\underbrace{[\lambda_{vw}]}_{\lambda^T} + \underbrace{[\boldsymbol{\beta}_{Unfiltered}]}_{B^{NT}}\underbrace{[\lambda_{Unfiltered}]}_{\lambda^{NT}}$$

$$\boldsymbol{r}_t = \underbrace{[\boldsymbol{\beta}_{Unfiltered}]}_{B^{NT}}\underbrace{[\lambda_{Unfiltered}]}_{\lambda^{NT}} + \underbrace{[\boldsymbol{\beta}_{vw}]}_{B^T}\underbrace{[f_{vw}]}_{f^T} + \underbrace{[\boldsymbol{\beta}_{Unfiltered}]}_{B^{NT}}\left(\underbrace{[f_{Unfiltered}]}_{f^{NT}} - \underbrace{[\mu_{Unfiltered}]}_{\mu^{NT}}\right) + \boldsymbol{e}_t$$

## IA.II Additional Simulations

### IA.II.1 Simulation for Useless Factor under Traded Factor Constriant

We simulate traded factors to emphasize the critical role of imposing restrictions on risk premia estimation. Lewellen et al. (2010) use artificial factors with zero means to demonstrate that, without such constraints, cross-sectional regressions prioritize fit, yielding a significantly nonzero simulated CSR$^2$, despite its theoretical value being zero. Extending their approach with marginal likelihood, our simulation highlights the necessity of these constraints.

Just as Lewellen et al. (2010)'s simulation study, we use data from empirical studies directly. Specifically, we use the same 25 ME-B/M portfolio and FF3 factors. We consider two kinds of useless factors. $\text{ul}_{1t}$ standard for a zero mean noisy factor, while $\text{ul}_{2t}$ also has zero mean, but correlated with return:

$$\text{ul}_{1t} = u_{1t}, u_{1t} \sim \mathcal{N}(0, (1\%)^2)$$
$$\text{ul}_{2t} = 0.5 * (\bar{r}_t - \bar{\bar{r}}) + u_{2t}, u_{2t} \sim \mathcal{N}(0, (1\%)^2)$$

where $\bar{r}_t$ is the cross-sectional average of test asset returns. Under this setup, these two factors should not have any pricing ability relative to other true factors. Under the imposed constraint, the model incorporating this factor is anticipated to underperform relative to the unconstrained model. Without this constraint, the model optimizes for the best fit by allowing for a nonzero risk premium deviation, thereby enhancing the marginal likelihood.

Table IA.1 shows the results for this simulation. The number is the average of 100 simulation runs. Each row represents the useless factor, which is $\text{ul}_{1t}$ or $\text{ul}_{2t}$. Panel A uses the estimation without the traded factor constraint, while Panel B imposes that constraint. Log ML reduced represents the logML without the chosen simulated factor, while log ML Full stands for the model with all 4 factors (FF3 factors plus one artificial factor). Panel C reports GLS CSR$^2$ for the reduced and full models, and Panel D reports the HJD.

All factors in Panel A exhibit higher marginal likelihoods when the corresponding

Table IA.1: **Evaluating Useless Factors under Traded Factor Constraint**

This table summarizes simulation results evaluating the impact of useless factors under the traded factor constraint. Panel A presents outcomes from the Bayesian model without the constraint, while Panel B incorporates it. The empirical dataset includes 25 ME-B/M portfolios, with FF3 factors and one simulated zero-mean factor. $\mathrm{ul}_{1t}$ represents a zero-mean factor uncorrelated with returns, and $\mathrm{ul}_{2t}$ is zero-mean but correlated with returns. "Log ML Reduced" denotes the log marginal likelihood of the model excluding the simulated factor, while "Log ML Full" corresponds to the full model. Panel C reports GLS $\mathrm{CSR}^2$ for the reduced and full models, and Panel D reports the Hansen-Jagannathan Distance (HJD). Results are averaged over 100 simulations.

| | log ML reduced | log ML Full | GLS reduced | GLS Full |
|---|---|---|---|---|
| | Panel A: Bayes No Constraint | | Paenl C: $\mathrm{CSR}^2$(%) | |
| $\mathrm{ul}_{1t}$ | 29604.0 | 29660.6 | 19.78 | 23.45 |
| $\mathrm{ul}_{2t}$ | 29605.0 | 29689.0 | 19.78 | 26.81 |
| | Panel B: Bayes With Constraint | | Panel D: HJD(%) | |
| $\mathrm{ul}_{1t}$ | 29392.6 | 29373.5 | 0.0373 | 0.0356 |
| $\mathrm{ul}_{2t}$ | 29377.2 | 29377.1 | 0.0373 | 0.0350 |

simulated factor is included, despite its zero mean. In contrast, Panel B shows a decline in marginal likelihood when a redundant factor is added, indicating no additional explanatory power. Across panels, marginal likelihood typically increases as constraints are relaxed. However, this does not imply that the unconstrained model outperforms the constrained one. The unconstrained model often prioritizes fit over theoretical alignment, diverging from the theoretical framework. Artificial factors should not enhance performance. Relaxing constraints can artificially inflate marginal likelihood, leading to misleading conclusions. To validate Lewellen et al. (2010)'s results, Panels C and D illustrate that adding a zero-mean, non-informative factor raises $\mathrm{CSR}^2$ while reducing the HJD, falsely suggesting improved pricing performance.

## IA.III Additional Empirical Results

### IA.III.1 The Role of Traded Factor Constraint

As suggested by Lewellen et al. (2010), we consider a large cross section of test assets and impose the traded factor constraint. This section shows how these suggestions help improve the empirical test. We consider all subsets of test assets and 10 traded factors as described in section 4.1. In this section, we do not include nontraded factors to emphasize the role of the traded factor constraint. The full set of factors and the subset of factors included in the return model are these 10 traded factors. We estimate this factor model specification under different sets of test assets.

Table IA.5 in the Appendix reports the estimation results of 10 traded factors and different panel stands for different sets of test assets. Within each panel, we report three types of estimation: Bayes without traded factor constraint, Bayes with that constraint, and WLS (without constraint). For each factor, we report the posterior mean for Bayes (mean for WLS) and 5%/ 95% credible interval (confidence interval for WLS).

Table IA.5 in the Appendix presents the estimation results for 10 traded factors. Each panel corresponds to a different set of test assets. Within each panel, we report results from three approaches: Bayesian without the traded factor constraint, Bayesian with the constraint, and WLS (without the constraint). These results show that imposing the traded factor constraint produces more stable estimates, which is to be expected. Without this constraint, the risk premia estimates vary more, even altering their statistical significance. For example, the CMA factor is largely insignificant across most test asset subsets. However, under the constraint, it appears slightly significant. Similar patterns emerge for BAB, IA, and ROE, where the unconstrained estimates suggest a lower or even negative effect, whereas the estimates indicate high significance under the constraint.

## IA.IV Additional Tables and Figures

Table IA.2: **Definition of Factors**

This table shows the factor explanation and sources. We include 18 factors, with 10 traded factors and 8 nontraded factors.

| **Factors** | **Explanations** | **Sources** |
|---|---|---|
| | **Traded Factors** | |
| MktRf | The excess return of the market portfolio | Sharpe (1964) |
| SMB | Small Minus Big | Fama and French (1993) |
| HML | High Minus Low (Value minus Growth) | Fama and French (1993) |
| RMW | Robust Minus Weak | Fama and French (2015) |
| CMA | Conservative Minus Aggressive | Fama and French (2015) |
| MOM | The momentum factor | Carhart (1997) |
| BAB | Betting Against Beta | Frazzini and Pedersen (2014) |
| QMJ | Quality Minus Junk | Asness et al. (2019) |
| IA | The investment factor | Hou et al. (2015) |
| ROE | The return on equity factor | Hou et al. (2015) |
| | **Nontraded Factors** | |
| IndProd | Industrial Production | Chen et al. (1986) |
| LIQ | Liquidity | Pastor and Stambaugh (2003) |
| LTY | Long Term Yield | Sweeney and Warga (1986) |
| M2 | M2 Seasonally Adjusted | Chan et al. (1996) |
| PCEDG | Personal Consumption Expenditures Durable | Yogo (2006) |
| PCEND | Personal Consumption Expenditures Non-Durable | Yogo (2006) |
| HKMcapital | Intermediary Capital Ratio and Risk Factor | He et al. (2017) |
| PEAR | Monthly presidential economic approval rating index | Chen et al. (2023) |

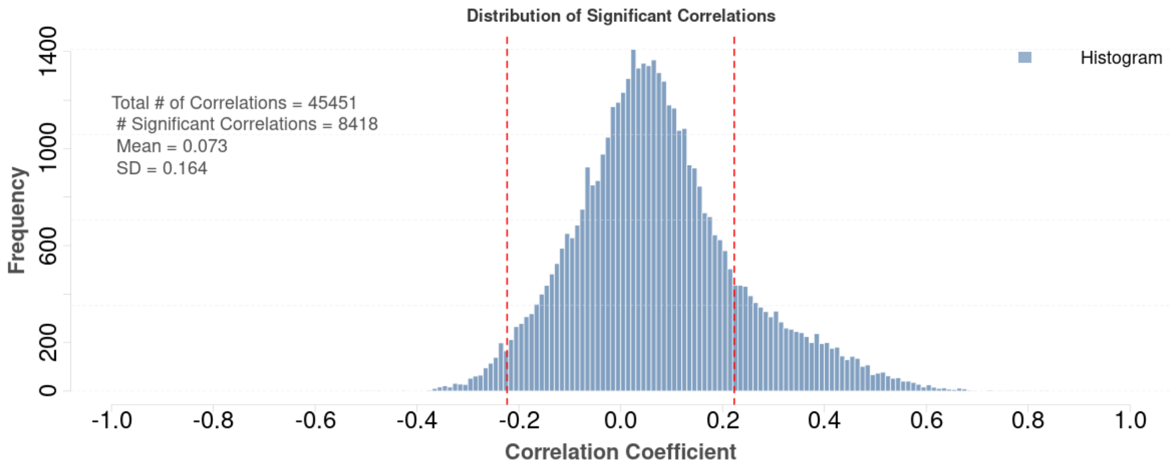


Figure IA.1: Residual Correlation Distribution

This figure shows the residual correlation distribution of 302 test assets regressed on our selected rank 1 model (in Table 6, with eight factors). The red dotted line indicates the cut-off for significance. Only 18% of correlations are significantly different from zero, according to Bonferroni multiple testing. This suggests that the diagonal assumption of residual covariance for test assets is not problematic.

Table IA.3: **Rejection Rate in Simulation**

This table shows the 5% rejection rate of the frequency of rejecting the null hypothesis $H_0 : \lambda = \lambda_{true}$ in simulation under the true model. There are 8 factors, including 6 traded factors and 2 nontraded factors. The parameters are calibrated from the true test assets and true factors. The number is calculated from 1000 simulation runs.

| | WLS | Bayes | BayesNC | WLS | Bayes | BayesNC |
|---|---|---|---|---|---|---|
| | 25 test assets | | | 100 test assets | | |
| MktRf | 0.055 | 0.046 | 0.076 | 0.050 | 0.041 | 0.353 |
| SMB | 0.052 | 0.060 | 0.066 | 0.056 | 0.043 | 0.103 |
| HML | 0.041 | 0.040 | 0.045 | 0.044 | 0.039 | 0.064 |
| RMW | 0.047 | 0.057 | 0.036 | 0.055 | 0.049 | 0.069 |
| CMA | 0.060 | 0.038 | 0.039 | 0.037 | 0.032 | 0.042 |
| UMD | 0.069 | 0.075 | 0.047 | 0.054 | 0.064 | 0.065 |
| PCEDG | 0.049 | 0.042 | 0.020 | 0.060 | 0.039 | 0.039 |
| HKMcapital | 0.069 | 0.041 | 0.042 | 0.050 | 0.043 | 0.152 |

Table IA.4: **Simulation Evidence: Model Comparison under Marginal Likelihood with Full Covariance Matrix**

This table summarizes simulation results comparing 1008 model specifications with 10 factors: 6 traded and 4 nontraded. The true DGP is a four-factor model (MktRf, SMB, HML, HKMcapital) with a full covariance matrix. Parameters are calibrated using the true test assets and factors. Metrics reported include log marginal likelihood, Cross-Sectional $R^2$ (%), average model rank by marginal likelihood, and Hansen-Jagannathan Distance (HJD %). NC denotes the Bayesian model without traded factor constraints. Simulations are conducted 100 times. Table 4 uses a diagonal covariance matrix for return generation, but this table uses a full covariance matrix for robustness checks. It shows that the diagonal assumption of the residual covariance matrix will not affect the model comparison even if the true DGP has a full covariance structure.

| Top 10 marginal likelihood models | logML | logMLNC | rank | rankNC | CSR$^2$ | HJD |
|---|---|---|---|---|---|---|
| MKTRF+SMB+HML+HKMcapital | **55314.9** | 55461.2 | **1.0** | 9.7 | 62.8 | 0.0167 |
| MKTRF+SMB+HML+PEAR+HKMcapital | 55253.7 | 55506.5 | 4.6 | 6.3 | 64.6 | 0.0161 |
| MKTRF+SMB+HML+UMD+HKMcapital | 55253.7 | 55528.4 | 4.7 | 3.5 | 64.4 | 0.0163 |
| MKTRF+SMB+HML+RMW+HKMcapital | 55253.3 | 55530.6 | 4.6 | 3.3 | 64.3 | 0.0162 |
| MKTRF+SMB+HML+Liq+HKMcapital | 55253.3 | 55529.2 | 4.7 | 3.3 | 64.2 | 0.0162 |
| MKTRF+SMB+HML+CMA+HKMcapital | 55253.1 | 55520.9 | 4.6 | 4.7 | 64.3 | 0.0162 |
| MKTRF+SMB+HML+PCEND+HKMcapital | 55252.6 | 55510.3 | 4.7 | 5.9 | 64.1 | 0.0162 |
| MKTRF+SMB+HML | 55220.8 | 55683.8 | 10.1 | 1.0 | 48.7 | 0.0196 |
| MKTRF+SMB+HML+UMD+PEAR+HKMcapital | 55192.3 | 55437.0 | 16.6 | 12.9 | 66.1 | 0.0157 |
| MKTRF+SMB+HML+CMA+PEAR+HKMcapital | 55192.0 | 55436.6 | 16.5 | 12.9 | 66.0 | 0.0156 |

Table IA.5 **Estimation for Traded Factors under Different Test Assets**

This table reports estimates from the traded-factor-only model using various test assets. We compare Bayesian models alongside WLS with and without traded factor constraints. The 95% credible (confidence) intervals are shown in parentheses.

| | MktRf | SMB | HML | RMW | CMA | UMD | BAB | QMJ | IA | ROE |
|---|---|---|---|---|---|---|---|---|---|---|
| | $5 \times 5$ Size and Book-to-Market | | | | | | | | | |
| Bayes No Constraint | 0.51 | -0.01 | 0.14 | 0.61 | 0.25 | 2.28 | 0.63 | 0.45 | 0.08 | 0.58 |
| | [ 0.30,0.74] | [-0.27,0.25] | [-0.13,0.45] | [ 0.19,1.05] | [-0.41,0.85] | [ 0.52,4.10] | [-0.61,1.93] | [-0.07,0.96] | [-0.61,0.71] | [-0.06,1.25] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.80 | 0.09 | 0.12 | 0.48 | 0.20 | 2.00 | 0.61 | 0.49 | -0.10 | 0.50 |
| | [ 0.39,1.22] | [-0.18,0.37] | [-0.17,0.41] | [ 0.06,0.89] | [-0.32,0.72] | [ 0.49,3.51] | [-0.35,1.58] | [ 0.05,0.92] | [-0.63,0.43] | [-0.06,1.06] |
| | 17 Industry | | | | | | | | | |
| Bayes No Constraint | 0.68 | -0.51 | 0.02 | 0.31 | 0.12 | 0.64 | -0.16 | 0.13 | 0.06 | 0.52 |
| | [ 0.38,0.99] | [-1.24,0.16] | [-0.39,0.40] | [-0.06,0.67] | [-0.24,0.47] | [-0.52,1.96] | [-1.25,0.86] | [-0.26,0.52] | [-0.30,0.46] | [-0.04,1.13] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.77 | 0.45 | 0.24 | 0.50 |
| | [ 0.29,1.15] | [-0.16,0.38] | [-0.11,0.44] | [ 0.19,0.61] | [ 0.03,0.42] | [-0.03,0.81] | [ 0.44,1.11] | [ 0.21,0.69] | [ 0.04,0.44] | [ 0.26,0.75] |
| WLS | 0.81 | -0.56 | 0.04 | 0.42 | -0.01 | 0.67 | -0.12 | 0.23 | -0.19 | 0.43 |
| | [ 0.40,1.23] | [-1.27,0.15] | [-0.40,0.47] | [-0.07,0.91] | [-0.49,0.47] | [-0.55,1.89] | [-1.13,0.90] | [-0.25,0.71] | [-0.73,0.35] | [-0.16,1.03] |
| | $5 \times 5$ Operating Profitability and Investment | | | | | | | | | |
| Bayes No Constraint | 0.62 | -0.18 | 0.20 | 0.34 | 0.25 | 0.38 | -0.22 | 0.09 | 0.18 | 0.27 |
| | [ 0.42,0.86] | [-0.97,0.53] | [-0.27,0.66] | [ 0.02,0.64] | [ 0.02,0.48] | [-0.66,1.41] | [-1.18,0.79] | [-0.34,0.54] | [-0.16,0.53] | [-0.54,1.02] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.76 | 0.18 | 0.39 | 0.45 | 0.31 | 0.35 | -0.24 | -0.14 | 0.05 | 0.10 |
| | [ 0.34,1.17] | [-0.47,0.84] | [-0.13,0.91] | [ 0.11,0.79] | [ 0.07,0.54] | [-0.68,1.39] | [-1.16,0.69] | [-0.57,0.30] | [-0.26,0.36] | [-0.65,0.85] |
| | $5 \times 5$ Size and Variance | | | | | | | | | |
| Bayes No Constraint | 0.43 | 0.03 | 0.53 | 0.23 | -0.16 | 0.64 | 0.59 | 0.40 | -0.15 | -0.14 |

Table IA.5 **continued from previous page**

| | MktRf | SMB | HML | RMW | CMA | UMD | BAB | QMJ | IA | ROE |
|---|---|---|---|---|---|---|---|---|---|---|
| | [ 0.21,0.67] | [-0.27,0.34] | [-0.45,1.58] | [-0.25,0.72] | [-0.82,0.52] | [-1.14,2.47] | [-0.42,1.55] | [-0.23,1.04] | [-0.86,0.55] | [-0.89,0.57] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.83 | 0.23 | -0.77 | 0.63 | -1.06 | 0.06 | 1.07 | 0.89 | -1.30 | -0.24 |
| | [ 0.41, 1.24] | [-0.08, 0.55] | [-1.69, 0.16] | [ 0.21, 1.05] | [-1.54,-0.58] | [-1.34, 1.45] | [ 0.18, 1.97] | [ 0.37, 1.41] | [-1.90,-0.70] | [-0.99, 0.50] |
| | | | | | $5 \times 5$ Size and Net Share Issues | | | | | |
| Bayes No Constraint | 0.47 | -0.18 | 0.80 | 0.26 | 0.45 | 2.43 | 0.54 | 0.38 | 0.25 | 0.15 |
| | [ 0.31,0.63] | [-0.41,0.05] | [ 0.25,1.34] | [-0.13,0.67] | [ 0.07,0.85] | [ 0.78,4.07] | [-0.34,1.46] | [-0.01,0.79] | [-0.22,0.75] | [-0.57,0.87] |
| Bayes With Constraint | 0.73 | 0.10 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.30,1.15] | [-0.16,0.39] | [-0.12,0.44] | [ 0.19,0.62] | [ 0.02,0.41] | [-0.02,0.83] | [ 0.44,1.10] | [ 0.24,0.69] | [ 0.02,0.43] | [ 0.26,0.74] |
| WLS | 0.78 | 0.02 | 0.54 | 0.22 | 0.37 | 2.21 | 0.75 | 0.28 | 0.17 | 0.49 |
| | [ 0.37,1.20] | [-0.26,0.31] | [-0.02,1.10] | [-0.16,0.60] | [ 0.05,0.70] | [ 0.89,3.52] | [ 0.05,1.45] | [-0.07,0.63] | [-0.21,0.56] | [-0.17,1.14] |
| | | | | | $5 \times 5$ Size and Accruals | | | | | |
| Bayes No Constraint | 0.46 | -0.10 | 0.01 | 0.06 | 0.25 | 1.10 | -0.23 | 0.24 | 0.19 | 0.10 |
| | [ 0.22,0.74] | [-0.35,0.18] | [-0.68,0.69] | [-0.47,0.60] | [-0.20,0.70] | [-0.48,2.74] | [-1.43,0.98] | [-0.21,0.69] | [-0.26,0.63] | [-0.59,0.81] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.77 | 0.04 | 0.13 | 0.10 | 0.32 | 0.80 | -0.64 | 0.18 | 0.19 | -0.14 |
| | [ 0.34,1.19] | [-0.26,0.33] | [-0.57,0.82] | [-0.38,0.58] | [-0.06,0.70] | [-0.75,2.34] | [-1.77,0.49] | [-0.25,0.61] | [-0.16,0.55] | [-0.84,0.56] |
| | | | | | $5 \times 5$ Size and Beta | | | | | |
| Bayes No Constraint | 0.47 | -0.08 | 0.43 | 0.18 | -0.11 | 1.02 | 0.39 | -0.09 | 0.06 | 0.01 |
| | [ 0.24,0.72] | [-0.35,0.20] | [-0.37,1.25] | [-0.18,0.56] | [-0.61,0.41] | [-0.81,2.86] | [-0.28,1.02] | [-0.59,0.40] | [-0.51,0.63] | [-0.78,0.77] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.82 | 0.11 | 0.01 | 0.15 | -0.38 | 1.13 | 0.67 | -0.06 | -0.24 | 0.16 |
| | [ 0.40,1.23] | [-0.18,0.40] | [-0.61,0.64] | [-0.22,0.52] | [-0.81,0.04] | [-0.14,2.40] | [ 0.10,1.23] | [-0.51,0.40] | [-0.75,0.27] | [-0.58,0.89] |

Table IA.5 **continued from previous page**

| | MktRf | SMB | HML | RMW | CMA | UMD | BAB | QMJ | IA | ROE |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | 5 × 5 Size and Momentum | | | | | |
| Bayes No Constraint | 0.42 | -0.10 | 0.81 | -0.13 | 0.54 | 0.70 | 0.25 | 0.06 | 0.74 | 0.38 |
| | [ 0.19,0.67] | [-0.36,0.18] | [-0.14,1.81] | [-0.63,0.41] | [-0.21,1.22] | [ 0.27,1.12] | [-0.68,1.17] | [-0.45,0.57] | [ 0.07,1.35] | [-0.20,0.93] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.86 | 0.06 | 0.52 | -0.51 | 0.92 | 0.50 | 0.18 | 0.03 | 0.93 | 0.58 |
| | [ 0.43, 1.28] | [-0.24, 0.35] | [-0.37, 1.40] | [-1.00,-0.01] | [ 0.22, 1.63] | [ 0.08, 0.91] | [-0.80, 1.15] | [-0.44, 0.49] | [ 0.26, 1.60] | [ 0.05, 1.11] |
| | | | | | 5 × 5 Size and Operating Profitability | | | | | |
| Bayes No Constraint | 0.48 | -0.05 | 0.47 | 0.41 | 0.33 | 1.41 | -0.12 | 0.04 | 0.44 | 0.39 |
| | [ 0.29,0.71] | [-0.29,0.21] | [-0.01,0.99] | [ 0.19,0.63] | [-0.20,0.88] | [-0.02,2.86] | [-1.28,1.02] | [-0.42,0.46] | [-0.20,1.04] | [-0.20,1.00] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.77 | 0.03 | 0.36 | 0.34 | 0.27 | 1.05 | -0.27 | -0.04 | 0.33 | 0.43 |
| | [ 0.36,1.18] | [-0.24,0.31] | [-0.10,0.83] | [ 0.11,0.58] | [-0.20,0.75] | [-0.22,2.32] | [-1.29,0.75] | [-0.47,0.38] | [-0.15,0.81] | [-0.08,0.93] |
| | | | | | 5 × 5 Size and Investment | | | | | |
| Bayes No Constraint | 0.47 | -0.22 | 0.99 | -0.28 | 0.23 | 0.07 | -0.39 | -0.13 | -0.11 | -0.12 |
| | [ 0.27,0.71] | [-0.53,0.12] | [ 0.09,1.72] | [-1.20,0.98] | [ 0.01,0.45] | [-1.91,2.28] | [-1.73,1.02] | [-0.73,0.57] | [-0.57,0.34] | [-0.84,0.63] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.80 | -0.23 | 1.36 | -1.11 | 0.15 | -0.92 | -0.70 | -0.48 | -0.30 | -0.19 |
| | [ 0.39, 1.21] | [-0.54, 0.09] | [ 0.66, 2.05] | [-1.98,-0.25] | [-0.06, 0.35] | [-2.44, 0.61] | [-1.77, 0.38] | [-1.01, 0.04] | [-0.66, 0.06] | [-0.72, 0.35] |
| | | | | | 5 × 5 Size and Short-Term Reversal | | | | | |
| Bayes No Constraint | 0.44 | -0.14 | 0.66 | 0.58 | 0.42 | -0.16 | 0.12 | 0.18 | 0.55 | 0.32 |
| | [ 0.22,0.69] | [-0.40,0.14] | [-0.15,1.54] | [-0.04,1.21] | [-0.21,1.07] | [-1.22,0.96] | [-0.66,0.86] | [-0.32,0.70] | [-0.19,1.29] | [-0.37,0.95] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |

Table IA.5 **continued from previous page**

| | MktRf | SMB | HML | RMW | CMA | UMD | BAB | QMJ | IA | ROE |
|---|---|---|---|---|---|---|---|---|---|---|
| WLS | 0.75 | 0.01 | -0.23 | 0.72 | 0.04 | -0.63 | 0.52 | 0.38 | 0.06 | 0.78 |
| | [ 0.33,1.16] | [-0.30,0.32] | [-1.23,0.77] | [ 0.02,1.42] | [-0.82,0.89] | [-1.77,0.51] | [-0.50,1.55] | [-0.13,0.89] | [-0.78,0.89] | [ 0.00,1.56] |
| | | | | | $5 \times 5$ Size and Long-Term Reversal | | | | | |
| Bayes No Constraint | 0.46 | 0.01 | 0.25 | 0.02 | 0.10 | 0.80 | 0.60 | 0.14 | 0.12 | 0.08 |
| | [ 0.24,0.70] | [-0.27,0.31] | [-0.21,0.75] | [-0.44,0.49] | [-0.31,0.49] | [-0.29,1.83] | [-0.31,1.56] | [-0.23,0.50] | [-0.30,0.54] | [-0.45,0.62] |
| Bayes With Constraint | 0.73 | 0.11 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.46 | 0.23 | 0.50 |
| | [ 0.27,1.17] | [-0.17,0.38] | [-0.12,0.46] | [ 0.19,0.61] | [ 0.03,0.40] | [ 0.01,0.83] | [ 0.43,1.11] | [ 0.23,0.67] | [ 0.04,0.42] | [ 0.25,0.74] |
| WLS | 0.81 | 0.23 | 0.13 | -0.13 | -0.01 | 1.13 | 1.07 | 0.06 | 0.05 | 0.22 |
| | [ 0.40,1.23] | [-0.09,0.55] | [-0.39,0.64] | [-0.62,0.35] | [-0.41,0.40] | [ 0.29,1.96] | [ 0.04,2.09] | [-0.31,0.43] | [-0.42,0.53] | [-0.43,0.88] |
| | | | | | All Portfolios | | | | | |
| Bayes No Constraint | 0.56 | -0.37 | 0.28 | 0.46 | 0.24 | 0.66 | 0.33 | 0.40 | 0.16 | 0.24 |
| | [ 0.47, 0.67] | [-0.49,-0.25] | [ 0.09, 0.47] | [ 0.29, 0.65] | [ 0.10, 0.39] | [ 0.31, 1.07] | [ 0.05, 0.61] | [ 0.25, 0.56] | [ 0.00, 0.33] | [ 0.03, 0.48] |
| Bayes With Constraint | 0.73 | 0.10 | 0.16 | 0.40 | 0.22 | 0.41 | 0.78 | 0.45 | 0.23 | 0.50 |
| | [ 0.30,1.17] | [-0.17,0.38] | [-0.11,0.44] | [ 0.20,0.60] | [ 0.04,0.41] | [-0.01,0.84] | [ 0.43,1.10] | [ 0.22,0.68] | [ 0.04,0.44] | [ 0.25,0.76] |
| WLS | 0.77 | 0.09 | 0.16 | 0.33 | 0.15 | 0.47 | 0.47 | 0.24 | 0.05 | 0.21 |
| | [ 0.36,1.18] | [-0.18,0.37] | [-0.14,0.46] | [ 0.09,0.57] | [-0.06,0.35] | [ 0.05,0.89] | [ 0.04,0.90] | [-0.03,0.51] | [-0.18,0.27] | [-0.19,0.60] |

Table IA.6: **Top 10 Models Ranked by Marginal Likelihood under 2 hyperparameter**

This table reports the top 10 models from all possible combinations of 18 factors, out of a total of 131,072 models under different training sample hyperparameters. The main results in Table 6 use a hyper-parameter of 4, while in this table, it is 2. The log marginal likelihood, Cross-Sectional $R^2$ (%), and Hansen-Jagannathan distance (HJD) are presented.

| Rank | Top 10 Model | logML | CSR$^2$ | HJD |
|---|---|---|---|---|
| 1 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ | 321589 | 49.03 | 0.0071 |
| 2 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + ROE | 321551 | 49.68 | 0.0071 |
| 3 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + HKMcapital | 321521 | 49.10 | 0.0071 |
| 4 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + PCEND | 321485 | 50.13 | 0.0070 |
| 5 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + ROE + HKMcapital | 321483 | 49.74 | 0.0071 |
| 6 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + PCEDG | 321466 | 49.10 | 0.0071 |
| 7 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + ROE + PCEND | 321430 | 50.42 | 0.0070 |
| 8 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + PEAR | 321428 | 49.55 | 0.0071 |
| 9 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + PCEND + HKMcapital | 321425 | 50.33 | 0.0070 |
| 10 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + IA | 321417 | 49.56 | 0.0071 |

Table IA.7: **Top 10 Models Ranked by Marginal Likelihood under 50 hyperparameter**

This table presents the top 10 models derived from all possible combinations of 18 factors, selected from a total of 131,072 models under varying training sample hyperparameters. The primary results in Table 6 use a hyperparameter value of 4, whereas this table employs a value of 50. Reported metrics include log marginal likelihood, Cross-Sectional $R^2$ (%), and Hansen-Jagannathan Distance (HJD).

| Rank | Top 10 Model | logML | $CSR^2$ | HJD |
|---|---|---|---|---|
| 1 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ | 320339 | 49.03 | 0.0071 |
| 2 | MktRf + SMB + HML + RMW + CMA + UMD + BAB | 320177 | 45.92 | 0.0073 |
| 3 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + ROE | 320085 | 49.68 | 0.0071 |
| 4 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + ROE | 320046 | 46.22 | 0.0072 |
| 5 | MktRf + SMB + HML + RMW + UMD + BAB + QMJ + IA | 320035 | 47.96 | 0.0072 |
| 6 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + PCEDG | 319960 | 49.10 | 0.0071 |
| 7 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + HKMcapital | 319906 | 49.10 | 0.0071 |
| 8 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + IA | 319895 | 49.56 | 0.0071 |
| 9 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + PCEND | 319875 | 50.13 | 0.0070 |
| 10 | MktRf + SMB + HML + RMW + CMA + UMD + BAB + QMJ + LTY | 319868 | 49.30 | 0.0071 |

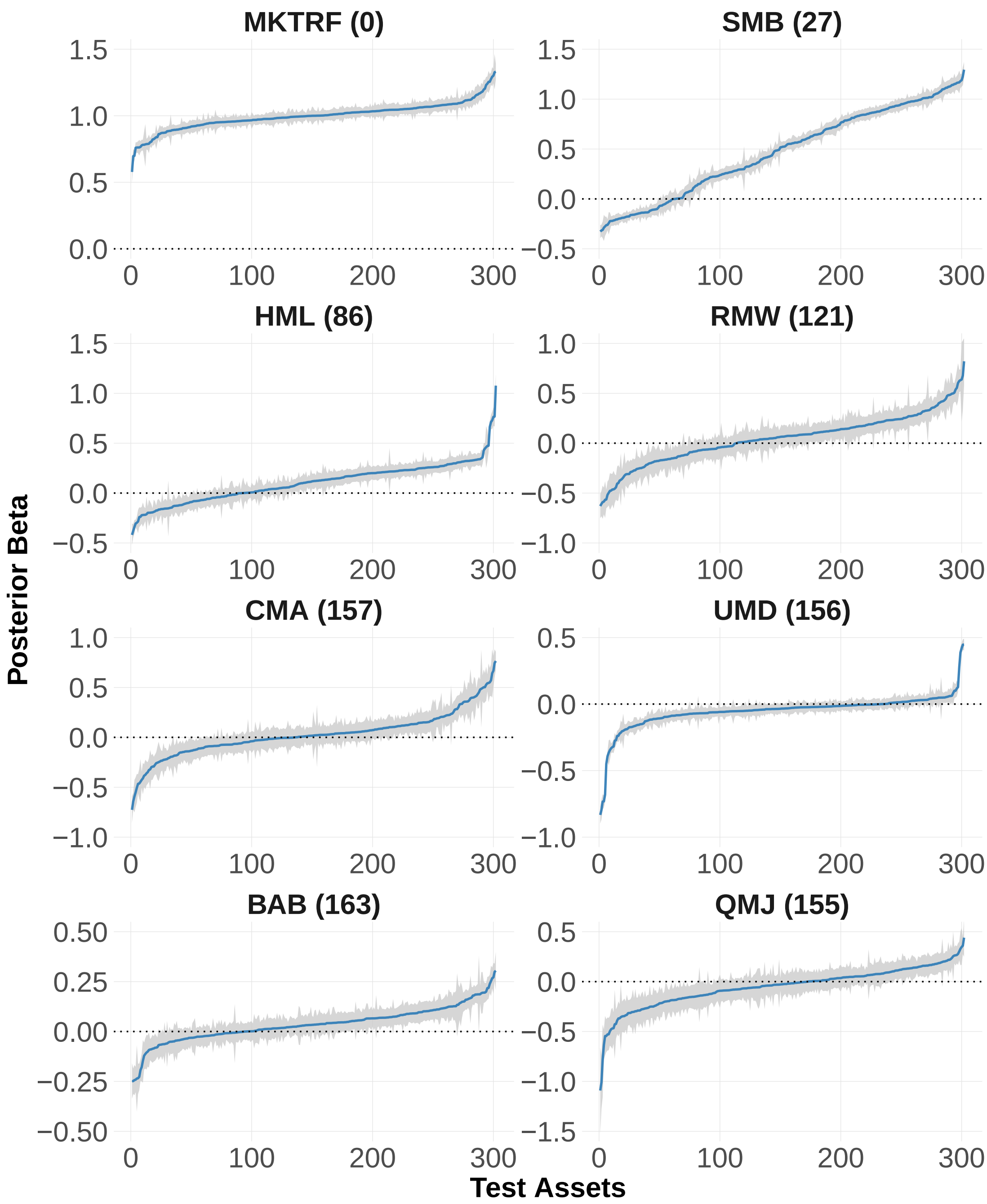


Figure IA.2: Posterior Cross-sectional Beta in the Top-1 model

This figure shows the posterior beta for each factor in the rank 1 model in Table 6 along with its 95% credible interval, represented by the grey shaded area. Beta is cross-sectionally sorted in ascending order. The number of betas for each factor that is not significantly different from 0 is in parentheses.

Table IA.8: **Out-of-Sample Model Performance: FF25**

Panel A reports annualized Sharpe Ratios (SR) and maximum drawdown (MDD) for tangency portfolios derived from seven-factor models, using out-of-sample (OOS) observations. Portfolio weights are estimated via Bayes and WLS methods. Weights are computed once using in-sample (INS) data (first half) and applied to OOS returns. The mean and covariance are taken from the subset of the full test assets estimation, but only apply to FF25 size-value portfolios. Panel B summarizes results for equal-weighted and mean-variance efficient portfolios.

| Panel A : Model-implied | | | | |
|---|---|---|---|---|
| | Bayes SR | WLS SR | Bayes MDD | WLS MDD |
| Rank 1 | 0.804 | 0.697 | 0.374 | 0.404 |
| Rank 2 | 0.824 | 0.627 | 0.368 | 0.434 |
| Rank 3 | 0.731 | 0.690 | 0.477 | 0.430 |
| CAPM | 0.559 | 0.557 | 0.473 | 0.474 |
| FF3 | 0.454 | 0.369 | 0.548 | 0.571 |
| FF5 | 0.779 | 0.759 | 0.357 | 0.338 |
| ALL | 0.753 | 0.623 | 0.361 | 0.382 |
| Panel B: EW and MVE | | | | |
| EW SR | MVE SR | | EW MDD | MVE MDD |
| 0.492 | 0.507 | | 0.490 | 0.493 |

Table IA.9: **Out-of-Sample Model Performance: IND17**

Panel A reports annualized Sharpe Ratios (SR) and maximum drawdown (MDD) for tangency portfolios derived from seven-factor models, using out-of-sample (OOS) observations. Portfolio weights are estimated via Bayes and WLS methods. Weights are computed once using in-sample (INS) data (first half) and applied to OOS returns. The mean and covariance are taken from the subset of the full test assets estimation, but only apply to 17 Industry portfolios. Panel B summarizes results for equal-weighted and mean-variance efficient portfolios.

| Panel A : Model-implied | | | | |
|---|---|---|---|---|
| | Bayes SR | WLS SR | Bayes MDD | WLS MDD |
| Rank 1 | 0.659 | 0.464 | 0.224 | 0.442 |
| Rank 2 | 0.647 | 0.503 | 0.231 | 0.434 |
| Rank 3 | 0.336 | 0.358 | 0.347 | 0.494 |
| CAPM | 0.591 | 0.593 | 0.492 | 0.489 |
| FF3 | 0.470 | 0.398 | 0.529 | 0.605 |
| FF5 | 0.536 | 0.463 | 0.394 | 0.434 |
| ALL | 0.787 | 0.536 | 0.224 | 0.389 |
| Panel B: EW and MVE | | | | |
| EW SR | MVE SR | | EW MDD | MVE MDD |
| 0.583 | 0.621 | | 0.501 | 0.461 |

## IA.V Model Specification with Zero-Beta Rate

This section proposes the model specification with a zero beta rate under the general setup. The return model in equation 5 is augmented with a zero beta rate $\lambda_0$,

$$\boldsymbol{r}_t = \mathbf{1}\lambda_0 + \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^T\boldsymbol{f}_t^T + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}) + \boldsymbol{e}_t \tag{IA.1}$$

and factors

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t \tag{IA.2}$$

The posterior distribution for the parameters of the model, given data from $t = 1, \ldots, T$, is proportional to the product of the likelihood and the prior. The return model likelihood is

$$p(\boldsymbol{r}_t \mid \lambda_0, \boldsymbol{f}_t^*, \boldsymbol{B}^T, \boldsymbol{B}^{NT}, \boldsymbol{\lambda}^{NT}, \boldsymbol{\Sigma}_e) = \prod_{t=1}^{T} \mathcal{N}(\boldsymbol{r}_t \mid \mathbf{1}\lambda_0 + \boldsymbol{B}^T\boldsymbol{f}_t^T + \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{B}^{NT}(\boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT}), \boldsymbol{\Sigma}_e),$$

where $\boldsymbol{\Sigma}_e = \text{diag}(\sigma_1^2, \ldots, \sigma_n^2)$, and the factor model likelihood is

$$p(\boldsymbol{f}_t^* \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}_u) = \prod_{t=1}^{T} \mathcal{N}(\boldsymbol{f}_t^* \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}_u).$$

The posterior distribution is proportional to the product of the two likelihoods and the prior on the parameters:

$$p(\lambda_0, \boldsymbol{B}^T, \boldsymbol{B}^{NT}, \boldsymbol{\lambda}^{NT}, \boldsymbol{\mu}, \boldsymbol{\Sigma}_e, \boldsymbol{\Sigma}_u).$$

We sample the resulting posterior distribution using MCMC methods. In this sampling, we consider the conditional distributions of blocks of parameters conditioned on the remaining parameters and sample these conditional distributions in sequence to produce a Markov chain whose invariant distribution is the posterior distribution (Chib, 2001). The conditional distributions of these different blocks of parameters are derived from the following formulations of the joint model. For updating $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_e$ we write the joint model as

$$\boldsymbol{r}_t = \mathbf{1}\lambda_0 + \boldsymbol{B}\tilde{\boldsymbol{f}}_t + \boldsymbol{e}_t$$

where

$$\tilde{\boldsymbol{f}}_t = \begin{pmatrix} \boldsymbol{f}_t^T \\ \boldsymbol{f}_t^{NT} - \boldsymbol{\mu}^{NT} + \boldsymbol{\lambda}^{NT} \end{pmatrix}$$

from which the conditional posterior distributions of $\boldsymbol{B}$ and $\boldsymbol{\Sigma}_e$ can be derived by Bayesian results for multivariate regression models. The updating of $\boldsymbol{\Sigma}_u$ arises easily from the factor model. For updating $\boldsymbol{\lambda}^{NT}$ we rewrite the full model now as

$$\boldsymbol{r}_t - \mathbf{1}\lambda_0 - \boldsymbol{B}\boldsymbol{f}_t + \boldsymbol{B}^{NT}\boldsymbol{\mu}^{NT} = \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} + \boldsymbol{e}_t$$

For updating $\lambda_0$ we rewrite the full model as

$$\boldsymbol{r}_t - \boldsymbol{B}\boldsymbol{f}_t + \boldsymbol{B}^{NT}\boldsymbol{\mu}^{NT} - \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} = \mathbf{1}\lambda_0 + \boldsymbol{e}_t$$

Finally, for updating $\boldsymbol{\mu}$ we write the joint model as

$$\boldsymbol{r}_t - \mathbf{1}\lambda_0 - \boldsymbol{B}^{NT}\boldsymbol{\lambda}^{NT} - \boldsymbol{B}\boldsymbol{f}_t = -\boldsymbol{B}^{NT}\boldsymbol{\mu}^{NT} + \boldsymbol{e}_t$$

and

$$\boldsymbol{f}_t^* = \boldsymbol{\mu} + \boldsymbol{u}_t$$

and then derive the conditional posterior of $\boldsymbol{\mu}$ in two steps, first with the return model likelihood and then with the factor model likelihood. The resulting MCMC algorithm is similar to previous cases, except that there is an additional block for sampling $\lambda_0$ from

$$\lambda_0 \sim \mathcal{N}(\widehat{\mu}_{\lambda_0}, \widehat{\sigma}^2_{\lambda_0}),$$

where, on letting $\tilde{\boldsymbol{R}} = \boldsymbol{R} - \tilde{\boldsymbol{F}}\boldsymbol{B} - \mathbf{1}'\boldsymbol{\lambda}^{NT'}\boldsymbol{B}^{NT'}$,

$$\widehat{\mu}_{\lambda_0} = \widehat{\sigma}^2_{\lambda_0}(\mathbf{1}'\boldsymbol{\Sigma}_e^{-1}\tilde{\boldsymbol{R}} + \widehat{\mu}_{0\lambda_0}\sigma^{-2}_{0\lambda_0}), \quad \widehat{\sigma}^{-2}_{\lambda_0} = \sigma^{-2}_{0\lambda_0} + \mathbf{1}'\boldsymbol{\Sigma}_e^{-1}\mathbf{1} \quad .$$

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